

All Theorems

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Section 2.1

Theorem 2.1.1 *If p is continuous on (a, b) , then the general solution of the homogeneous equation*

$$y' + p(x)y = 0 \quad (2.1.12)$$

on (a, b) is

$$y = ce^{-P(x)},$$

where

$$P(x) = \int p(x) dx \quad (2.1.13)$$

is any antiderivative of p on (a, b) ; that is,

$$P'(x) = p(x), \quad a < x < b. \quad (2.1.14)$$

Theorem 2.1.2 *Suppose p and f are continuous on an open interval (a, b) , and let y_1 be any nontrivial solution of the complementary equation*

$$y' + p(x)y = 0$$

on (a, b) . Then:

(a) The general solution of the nonhomogeneous equation

$$y' + p(x)y = f(x) \quad (2.1.31)$$

on (a, b) is

$$y = y_1(x) \left(c + \int f(x)/y_1(x) dx \right). \quad (2.1.32)$$

(b) If x_0 is an arbitrary point in (a, b) and y_0 is an arbitrary real number, then the initial value problem

$$y' + p(x)y = f(x), \quad y(x_0) = y_0$$

has the unique solution

$$y = y_1(x) \left(\frac{y_0}{y_1(x_0)} + \int_{x_0}^x \frac{f(t)}{y_1(t)} dt \right)$$

on (a, b) .

Section 2.2

Theorem 2.2.1 Suppose $g = g(x)$ is continuous on (a, b) and $h = h(y)$ is continuous on (c, d) . Let G be an antiderivative of g on (a, b) and let H be an antiderivative of h on (c, d) . Let x_0 be an arbitrary point in (a, b) , let y_0 be a point in (c, d) such that $h(y_0) \neq 0$, and define

$$c = H(y_0) - G(x_0). \quad (2.2.10)$$

Then there's a function $y = y(x)$ defined on some open interval (a_1, b_1) , where $a \leq a_1 < x_0 < b_1 \leq b$, such that $y(x_0) = y_0$ and

$$H(y) = G(x) + c \quad (2.2.11)$$

for $a_1 < x < b_1$. Therefore y is a solution of the initial value problem

$$h(y)y' = g(x), \quad y(x_0) = y_0. \quad (2.2.12)$$

Section 2.3

Theorem 2.3.1

(a) If f is continuous on an open rectangle

$$R : \{a < x < b, c < y < d\}$$

that contains (x_0, y_0) then the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0 \quad (2.3.1)$$

has at least one solution on some open subinterval of (a, b) that contains x_0 .

(b) If both f and f_y are continuous on R then (2.3.1) has a unique solution on some open subinterval of (a, b) that contains x_0 .

Section 2.5

Theorem 2.5.1 *If $F = F(x, y)$ has continuous partial derivatives F_x and F_y , then*

$$F(x, y) = c \quad (c=\text{constant}), \quad (2.5.6)$$

is an implicit solution of the differential equation

$$F_x(x, y) dx + F_y(x, y) dy = 0. \quad (2.5.7)$$

Theorem 2.5.2 [The Exactness Condition] *Suppose M and N are continuous and have continuous partial derivatives M_y and N_x on an open rectangle R . Then*

$$M(x, y) dx + N(x, y) dy = 0$$

is exact on R if and only if

$$M_y(x, y) = N_x(x, y) \quad (2.5.12)$$

for all (x, y) in R .

Section 2.6

Theorem 2.6.1 *Let M , N , M_y , and N_x be continuous on an open rectangle R . Then:*

(a) If $(M_y - N_x)/N$ is independent of y on R and we define

$$p(x) = \frac{M_y - N_x}{N}$$

then

$$\mu(x) = \pm e^{\int p(x) dx} \quad (2.6.10)$$

is an integrating factor for

$$M(x, y) dx + N(x, y) dy = 0 \quad (2.6.11)$$

on R .

(b) If $(N_x - M_y)/M$ is independent of x on R and we define

$$q(y) = \frac{N_x - M_y}{M},$$

then

$$\mu(y) = \pm e^{\int q(y) dy} \quad (2.6.12)$$

is an integrating factor for (2.6.11) on R .

Section 5.1

Theorem 5.1.1 Suppose p and q are continuous on an open interval (a, b) , let x_0 be any point in (a, b) , and let k_0 and k_1 be arbitrary real numbers. Then the initial value problem

$$y'' + p(x)y' + q(x)y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

has a unique solution on (a, b) .

Theorem 5.1.2 If y_1 and y_2 are solutions of the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0 \tag{5.1.18}$$

on (a, b) , then any linear combination

$$y = c_1y_1 + c_2y_2 \tag{5.1.19}$$

of y_1 and y_2 is also a solution of (5.1.18) on (a, b) .

Theorem 5.1.3 Suppose p and q are continuous on (a, b) . Then a set $\{y_1, y_2\}$ of solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{5.1.20}$$

on (a, b) is a fundamental set if and only if $\{y_1, y_2\}$ is linearly independent on (a, b) .

Theorem 5.1.4 Suppose p and q are continuous on (a, b) , let y_1 and y_2 be solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{5.1.27}$$

on (a, b) , and define

$$W = y_1y_2' - y_1'y_2. \tag{5.1.28}$$

Let x_0 be any point in (a, b) . Then

$$W(x) = W(x_0)e^{-\int_{x_0}^x p(t) dt}, \quad a < x < b. \tag{5.1.29}$$

Therefore either W has no zeros in (a, b) or $W \equiv 0$ on (a, b) .

Theorem 5.1.5 Suppose p and q are continuous on an open interval (a, b) , let y_1 and y_2 be solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{5.1.32}$$

on (a, b) , and let $W = y_1y_2' - y_1'y_2$. Then y_1 and y_2 are linearly independent on (a, b) if and only if W has no zeros on (a, b) .

Theorem 5.1.6 Suppose p and q are continuous on an open interval (a, b) and let y_1 and y_2 be solutions of

$$y'' + p(x)y' + q(x)y = 0 \tag{5.1.34}$$

on (a, b) . Then the following statements are equivalent; that is, they are either all true or all false.

(a) The general solution of (5.1.34) on (a, b) is $y = c_1y_1 + c_2y_2$.

(b) $\{y_1, y_2\}$ is a fundamental set of solutions of (5.1.34) on (a, b) .

- (c) $\{y_1, y_2\}$ is linearly independent on (a, b) .
 (d) The Wronskian of $\{y_1, y_2\}$ is nonzero at some point in (a, b) .
 (e) The Wronskian of $\{y_1, y_2\}$ is nonzero at all points in (a, b) .

Theorem 5.1.7 Suppose c is in (a, b) and α and β are real numbers, not both zero. Under the assumptions of Theorem 5.1.6, suppose y_1 and y_2 are solutions of (5.1.34) such that

$$\alpha y_1(c) + \beta y_1'(c) = 0 \quad \text{and} \quad \alpha y_2(c) + \beta y_2'(c) = 0. \quad (5.1.35)$$

Then $\{y_1, y_2\}$ isn't linearly independent on (a, b) .

Section 5.2

Theorem 5.2.1 Let $a \neq 0$ and let $p(r) = ar^2 + br + c$ be the characteristic polynomial of

$$ay'' + by' + cy = 0. \quad (5.2.20)$$

Then:

- (a) If $p(r) = 0$ has distinct real roots r_1 and r_2 , then the general solution of (5.2.20) is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

- (b) If $p(r) = 0$ has a repeated root r_1 , then the general solution of (5.2.20) is

$$y = e^{r_1 x} (c_1 + c_2 x).$$

- (c) If $p(r) = 0$ has complex conjugate roots $r_1 = \lambda + i\omega$ and $r_2 = \lambda - i\omega$ (where $\omega > 0$), then the general solution of (5.2.20) is

$$y = e^{\lambda x} (c_1 \cos \omega x + c_2 \sin \omega x).$$

Section 5.3

Theorem 5.3.1 Suppose p , q , and f are continuous on an open interval (a, b) , let x_0 be any point in (a, b) , and let k_0 and k_1 be arbitrary real numbers. Then the initial value problem

$$y'' + p(x)y' + q(x)y = f(x), \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

has a unique solution on (a, b) .

Theorem 5.3.2 Suppose p , q , and f are continuous on (a, b) . Let y_p be a particular solution of

$$y'' + p(x)y' + q(x)y = f(x) \quad (5.3.3)$$

on (a, b) , and let $\{y_1, y_2\}$ be a fundamental set of solutions of the complementary equation

$$y'' + p(x)y' + q(x)y = 0 \quad (5.3.4)$$

on (a, b) . Then y is a solution of (5.3.3) on (a, b) if and only if

$$y = y_p + c_1 y_1 + c_2 y_2, \quad (5.3.5)$$

where c_1 and c_2 are constants.

Theorem 5.3.3 [*The Principle of Superposition*] Suppose y_{p_1} is a particular solution of

$$y'' + p(x)y' + q(x)y = f_1(x)$$

on (a, b) and y_{p_2} is a particular solution of

$$y'' + p(x)y' + q(x)y = f_2(x)$$

on (a, b) . Then

$$y_p = y_{p_1} + y_{p_2}$$

is a particular solution of

$$y'' + p(x)y' + q(x)y = f_1(x) + f_2(x)$$

on (a, b) .

Section 5.5

Theorem 5.5.1 Suppose $a \neq 0$, ω is a positive number, and P and Q are polynomials, at least one nonzero. Let k be the larger of the degrees of P and Q . Then the equation

$$ay'' + by' + cy = P(x) \cos \omega x + Q(x) \sin \omega x$$

has a particular solution

$$y_p = A(x) \cos \omega x + B(x) \sin \omega x, \quad (5.5.3)$$

where

$$A(x) = A_0 + A_1x + \cdots + A_kx^k \quad \text{and} \quad B(x) = B_0 + B_1x + \cdots + B_kx^k,$$

provided that $\cos \omega x$ and $\sin \omega x$ are not solutions of the complementary equation. The equation

$$a(y'' + \omega^2 y) = P(x) \cos \omega x + Q(x) \sin \omega x$$

(for which $\cos \omega x$ and $\sin \omega x$ are solutions of the complementary equation) has a particular solution of the form (5.5.3), where

$$A(x) = A_0x + A_1x^2 + \cdots + A_kx^{k+1} \quad \text{and} \quad B(x) = B_0x + B_1x^2 + \cdots + B_kx^{k+1}.$$

Theorem 5.5.2 If

$$y_p = A(x) \cos \omega x + B(x) \sin \omega x,$$

where $A(x)$ and $B(x)$ are polynomials with coefficients $A_0, \dots, A_k$ and $B_0, \dots, B_k$, then the polynomials multiplying $\sin \omega x$ in

$$y'_p, \quad y''_p, \quad ay''_p + by'_p + cy_p \quad \text{and} \quad y''_p + \omega^2 y_p$$

can be obtained by replacing $A_0, \dots, A_k$ by $B_0, \dots, B_k$ and $B_0, \dots, B_k$ by $-A_0, \dots, -A_k$ in the corresponding polynomials multiplying $\cos \omega x$.

Section 6.2

Theorem 6.2.1 Suppose $m, k, c, F_0 > 0$ and $\omega \geq 0$. Consider the amplitude R of the steady state component of the solution of

$$my'' + cy' + ky = F_0 \cos \omega t$$

as a function of ω .

(a) If $c \geq \sqrt{2mk}$, the maximum amplitude is $R_{\max} = F_0/k$ and it's attained when $\omega = \omega_{\max} = 0$.

(b) If $c < \sqrt{2mk}$, the maximum amplitude is

$$R_{\max} = \frac{2mF_0}{c\sqrt{4mk - c^2}}, \quad (6.2.20)$$

and it's attained when

$$\omega = \omega_{\max} = \sqrt{\frac{k}{m} \left(1 - \frac{c^2}{2km} \right)}. \quad (6.2.21)$$

Section 6.4

Theorem 6.4.1 The position vector of an object moving under a central force sweeps out equal areas in equal times. More precisely, if $t_1 < t_2$, the signed area swept from $\theta(t_1)$ to $\theta(t_2)$ (Figure 6.4.3) is given by

$$A = \frac{h(t_2 - t_1)}{2},$$

where $h = r^2\theta'$, which we have shown to be constant.

Section 7.1

Theorem 7.1.2 For any power series

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n,$$

exactly one of these statements is true:

- (i) The power series converges only for $x = x_0$.
- (ii) The power series converges for all values of x .
- (iii) There's a positive number R such that the power series converges if $|x - x_0| < R$ and diverges if $|x - x_0| > R$.

Theorem 7.1.3 Suppose there's an integer N such that $a_n \neq 0$ if $n \geq N$ and

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = L,$$

where $0 \leq L \leq \infty$. Then the radius of convergence of $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ is $R = 1/L$, which should be interpreted to mean that $R = 0$ if $L = \infty$, or $R = \infty$ if $L = 0$.

Theorem 7.1.4 *A power series*

$$f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$$

with positive radius of convergence R has derivatives of all orders in its open interval of convergence, and successive derivatives can be obtained by repeatedly differentiating term by term; that is,

$$f'(x) = \sum_{n=1}^{\infty} n a_n(x - x_0)^{n-1}, \quad (7.1.6)$$

$$f''(x) = \sum_{n=2}^{\infty} n(n-1) a_n(x - x_0)^{n-2}, \quad (7.1.7)$$

$$\vdots$$

$$f^{(k)}(x) = \sum_{n=k}^{\infty} n(n-1) \cdots (n-k+1) a_n(x - x_0)^{n-k}. \quad (7.1.8)$$

Moreover, all of these series have the same radius of convergence R .

Theorem 7.1.5 *If the power series*

$$f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$$

has a positive radius of convergence, then

$$a_n = \frac{f^{(n)}(x_0)}{n!}; \quad (7.1.9)$$

that is, $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ is the Taylor series of f about x_0 .

Theorem 7.1.6

(a) *If*

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n = \sum_{n=0}^{\infty} b_n(x - x_0)^n$$

for all x in an open interval that contains x_0 , then $a_n = b_n$ for $n = 0, 1, 2, \dots$

(b) *If*

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n = 0$$

for all x in an open interval that contains x_0 , then $a_n = 0$ for $n = 0, 1, 2, \dots$

Section 7.2

Theorem 7.2.1 Suppose P_0 , P_1 , and P_2 are polynomials with no common factor and P_0 isn't identically zero. Let x_0 be a point such that $P_0(x_0) \neq 0$, and let ρ be the distance from x_0 to the nearest zero of P_0 in the complex plane. (If P_0 is constant, then $\rho = \infty$.) Then every solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0 \quad (7.2.5)$$

can be represented by a power series

$$y = \sum_{n=0}^{\infty} a_n(x - x_0)^n \quad (7.2.6)$$

that converges at least on the open interval $(x_0 - \rho, x_0 + \rho)$. (If P_0 is nonconstant, so that ρ is necessarily finite, then the open interval of convergence of (7.2.6) may be larger than $(x_0 - \rho, x_0 + \rho)$. If P_0 is constant then $\rho = \infty$ and $(x_0 - \rho, x_0 + \rho) = (-\infty, \infty)$.)

Theorem 7.2.2 The coefficients $\{a_n\}$ in any solution $y = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ of

$$\left(1 + \alpha(x - x_0)^2\right)y'' + \beta(x - x_0)y' + \gamma y = 0 \quad (7.2.23)$$

satisfy the recurrence relation

$$a_{n+2} = -\frac{p(n)}{(n+2)(n+1)}a_n, \quad n \geq 0, \quad (7.2.24)$$

where

$$p(n) = \alpha n(n-1) + \beta n + \gamma. \quad (7.2.25)$$

Moreover, the coefficients of the even and odd powers of $x - x_0$ can be computed separately as

$$a_{2m+2} = -\frac{p(2m)}{(2m+2)(2m+1)}a_{2m}, \quad m \geq 0 \quad (7.2.26)$$

and

$$a_{2m+3} = -\frac{p(2m+1)}{(2m+3)(2m+2)}a_{2m+1}, \quad m \geq 0, \quad (7.2.27)$$

where a_0 and a_1 are arbitrary.

Section 7.4

Theorem 7.4.3 Suppose the roots of the indicial equation

$$ar(r-1) + br + c = 0 \quad (7.4.11)$$

are r_1 and r_2 . Then the general solution of the Euler equation

$$ax^2y'' + bxy' + cy = 0 \quad (7.4.12)$$

on $(0, \infty)$ is

$$\begin{aligned} y &= c_1x^{r_1} + c_2x^{r_2} \text{ if } r_1 \text{ and } r_2 \text{ are distinct real numbers;} \\ y &= x^{r_1}(c_1 + c_2 \ln x) \text{ if } r_1 = r_2; \\ y &= x^\lambda [c_1 \cos(\omega \ln x) + c_2 \sin(\omega \ln x)] \text{ if } r_1, r_2 = \lambda \pm i\omega \text{ with } \omega > 0. \end{aligned}$$

Section 7.5

Theorem 7.5.1 *Let*

$$Ly = x^2(\alpha_0 + \alpha_1x + \alpha_2x^2)y'' + x(\beta_0 + \beta_1x + \beta_2x^2)y' + (\gamma_0 + \gamma_1x + \gamma_2x^2)y,$$

and define

$$\begin{aligned} p_0(r) &= \alpha_0r(r-1) + \beta_0r + \gamma_0, \\ p_1(r) &= \alpha_1r(r-1) + \beta_1r + \gamma_1, \\ p_2(r) &= \alpha_2r(r-1) + \beta_2r + \gamma_2. \end{aligned}$$

Suppose the series

$$y = \sum_{n=0}^{\infty} a_n x^{n+r} \quad (7.5.3)$$

converges on $(0, \rho)$. Then

$$Ly = \sum_{n=0}^{\infty} b_n x^{n+r} \quad (7.5.4)$$

on $(0, \rho)$, where

$$\begin{aligned} b_0 &= p_0(r)a_0, \\ b_1 &= p_0(r+1)a_1 + p_1(r)a_0, \\ b_n &= p_0(n+r)a_n + p_1(n+r-1)a_{n-1} + p_2(n+r-2)a_{n-2}, \quad n \geq 2. \end{aligned} \quad (7.5.5)$$

Theorem 7.5.2 *Let*

$$Ly = x^2(\alpha_0 + \alpha_1x + \alpha_2x^2)y'' + x(\beta_0 + \beta_1x + \beta_2x^2)y' + (\gamma_0 + \gamma_1x + \gamma_2x^2)y,$$

where $\alpha_0 \neq 0$, and define

$$\begin{aligned} p_0(r) &= \alpha_0r(r-1) + \beta_0r + \gamma_0, \\ p_1(r) &= \alpha_1r(r-1) + \beta_1r + \gamma_1, \\ p_2(r) &= \alpha_2r(r-1) + \beta_2r + \gamma_2. \end{aligned}$$

Suppose r is a real number such that $p_0(n+r)$ is nonzero for all positive integers n . Define

$$\begin{aligned} a_0(r) &= 1, \\ a_1(r) &= -\frac{p_1(r)}{p_0(r+1)}, \\ a_n(r) &= -\frac{p_1(n+r-1)a_{n-1}(r) + p_2(n+r-2)a_{n-2}(r)}{p_0(n+r)}, \quad n \geq 2. \end{aligned} \quad (7.5.12)$$

Then the Frobenius series

$$y(x, r) = x^r \sum_{n=0}^{\infty} a_n(r) x^n \quad (7.5.13)$$

converges and satisfies

$$Ly(x, r) = p_0(r)x^r \quad (7.5.14)$$

on the interval $(0, \rho)$, where ρ is the distance from the origin to the nearest zero of $A(x) = \alpha_0 + \alpha_1x + \alpha_2x^2$ in the complex plane. (If A is constant, then $\rho = \infty$.)

Theorem 7.5.3 Let L and $\{a_n(r)\}$ be as in Theorem 7.5.2, and suppose the indicial equation $p_0(r) = 0$ of $Ly = 0$ has real roots r_1 and r_2 , where $r_1 \geq r_2$. Then

$$y_1(x) = y(x, r_1) = x^{r_1} \sum_{n=0}^{\infty} a_n(r_1) x^n$$

is a Frobenius solution of $Ly = 0$. Moreover, if $r_1 - r_2$ isn't an integer then

$$y_2(x) = y(x, r_2) = x^{r_2} \sum_{n=0}^{\infty} a_n(r_2) x^n$$

is also a Frobenius solution of $Ly = 0$, and $\{y_1, y_2\}$ is a fundamental set of solutions.

Section 7.6

Theorem 7.6.1 Let

$$Ly = x^2(\alpha_0 + \alpha_1 x + \alpha_2 x^2)y'' + x(\beta_0 + \beta_1 x + \beta_2 x^2)y' + (\gamma_0 + \gamma_1 x + \gamma_2 x^2)y, \quad (7.6.2)$$

where $\alpha_0 \neq 0$ and define

$$\begin{aligned} p_0(r) &= \alpha_0 r(r-1) + \beta_0 r + \gamma_0, \\ p_1(r) &= \alpha_1 r(r-1) + \beta_1 r + \gamma_1, \\ p_2(r) &= \alpha_2 r(r-1) + \beta_2 r + \gamma_2. \end{aligned}$$

Suppose r is a real number such that $p_0(n+r)$ is nonzero for all positive integers n , and define

$$\begin{aligned} a_0(r) &= 1, \\ a_1(r) &= -\frac{p_1(r)}{p_0(r+1)}, \\ a_n(r) &= -\frac{p_1(n+r-1)a_{n-1}(r) + p_2(n+r-2)a_{n-2}(r)}{p_0(n+r)}, \quad n \geq 2. \end{aligned}$$

Then the Frobenius series

$$y(x, r) = x^r \sum_{n=0}^{\infty} a_n(r) x^n \quad (7.6.3)$$

satisfies

$$Ly(x, r) = p_0(r)x^r. \quad (7.6.4)$$

Moreover,

$$\frac{\partial y}{\partial r}(x, r) = y(x, r) \ln x + x^r \sum_{n=1}^{\infty} a'_n(r) x^n, \quad (7.6.5)$$

and

$$L\left(\frac{\partial y}{\partial r}(x, r)\right) = p'_0(r)x^r + x^r p_0(r) \ln x. \quad (7.6.6)$$

Theorem 7.6.2 *Let L be as in Theorem 7.6.1 and suppose the indicial equation $p_0(r) = 0$ has a repeated real root r_1 . Then*

$$y_1(x) = y(x, r_1) = x^{r_1} \sum_{n=0}^{\infty} a_n(r_1) x^n$$

and

$$y_2(x) = \frac{\partial y}{\partial r}(x, r_1) = y_1(x) \ln x + x^{r_1} \sum_{n=1}^{\infty} a'_n(r_1) x^n \quad (7.6.8)$$

form a fundamental set of solutions of $Ly = 0$.

Section 7.7

Theorem 7.7.1 *Let*

$$Ly = x^2(\alpha_0 + \alpha_1 x)y'' + x(\beta_0 + \beta_1 x)y' + (\gamma_0 + \gamma_1 x)y,$$

where $\alpha_0 \neq 0$, and define

$$\begin{aligned} p_0(r) &= \alpha_0 r(r-1) + \beta_0 r + \gamma_0, \\ p_1(r) &= \alpha_1 r(r-1) + \beta_1 r + \gamma_1. \end{aligned}$$

Suppose r is a real number such that $p_0(n+r)$ is nonzero for all positive integers n , and define

$$\begin{aligned} a_0(r) &= 1, \\ a_n(r) &= -\frac{p_1(n+r-1)}{p_0(n+r)} a_{n-1}(r), \quad n \geq 1. \end{aligned} \quad (7.7.2)$$

Let r_1 and r_2 be the roots of the indicial equation $p_0(r) = 0$, and suppose $r_1 = r_2 + k$, where k is a positive integer. Then

$$y_1 = x^{r_1} \sum_{n=0}^{\infty} a_n(r_1) x^n$$

is a Frobenius solution of $Ly = 0$. Moreover, if we define

$$\begin{aligned} a_0(r_2) &= 1, \\ a_n(r_2) &= -\frac{p_1(n+r_2-1)}{p_0(n+r_2)} a_{n-1}(r_2), \quad 1 \leq n \leq k-1, \end{aligned} \quad (7.7.3)$$

and

$$C = -\frac{p_1(r_1-1)}{k\alpha_0} a_{k-1}(r_2), \quad (7.7.4)$$

then

$$y_2 = x^{r_2} \sum_{n=0}^{k-1} a_n(r_2) x^n + C \left(y_1 \ln x + x^{r_1} \sum_{n=1}^{\infty} a'_n(r_1) x^n \right) \quad (7.7.5)$$

is also a solution of $Ly = 0$, and $\{y_1, y_2\}$ is a fundamental set of solutions.

Theorem 7.7.2 *Let*

$$Ly = x^2(\alpha_0 + \alpha_2 x^2)y'' + x(\beta_0 + \beta_2 x^2)y' + (\gamma_0 + \gamma_2 x^2)y,$$

where $\alpha_0 \neq 0$, and define

$$p_0(r) = \alpha_0 r(r-1) + \beta_0 r + \gamma_0,$$

$$p_2(r) = \alpha_2 r(r-1) + \beta_2 r + \gamma_2.$$

Suppose r is a real number such that $p_0(2m+r)$ is nonzero for all positive integers m , and define

$$\begin{aligned} a_0(r) &= 1, \\ a_{2m}(r) &= -\frac{p_2(2m+r-2)}{p_0(2m+r)} a_{2m-2}(r), \quad m \geq 1. \end{aligned} \quad (7.7.22)$$

Let r_1 and r_2 be the roots of the indicial equation $p_0(r) = 0$, and suppose $r_1 = r_2 + 2k$, where k is a positive integer. Then

$$y_1 = x^{r_1} \sum_{m=0}^{\infty} a_{2m}(r_1) x^{2m}$$

is a Frobenius solution of $Ly = 0$. Moreover, if we define

$$\begin{aligned} a_0(r_2) &= 1, \\ a_{2m}(r_2) &= -\frac{p_2(2m+r_2-2)}{p_0(2m+r_2)} a_{2m-2}(r_2), \quad 1 \leq m \leq k-1 \end{aligned}$$

and

$$C = -\frac{p_2(r_1-2)}{2k\alpha_0} a_{2k-2}(r_2), \quad (7.7.23)$$

then

$$y_2 = x^{r_2} \sum_{m=0}^{k-1} a_{2m}(r_2) x^{2m} + C \left(y_1 \ln x + x^{r_1} \sum_{m=1}^{\infty} a'_{2m}(r_1) x^{2m} \right) \quad (7.7.24)$$

is also a solution of $Ly = 0$, and $\{y_1, y_2\}$ is a fundamental set of solutions.

Section 8.1

Theorem 8.1.2 [Linearity Property] Suppose $\mathcal{L}(f_i)$ is defined for $s > s_i$, $1 \leq i \leq n$. Let s_0 be the largest of the numbers $s_1, s_2, \dots, s_n$, and let $c_1, c_2, \dots, c_n$ be constants. Then

$$\mathcal{L}(c_1 f_1 + c_2 f_2 + \dots + c_n f_n) = c_1 \mathcal{L}(f_1) + c_2 \mathcal{L}(f_2) + \dots + c_n \mathcal{L}(f_n) \text{ for } s > s_0.$$

Theorem 8.1.3 [First Shifting Theorem] If

$$F(s) = \int_0^{\infty} e^{-st} f(t) dt \quad (8.1.10)$$

is the Laplace transform of $f(t)$ for $s > s_0$, then $F(s-a)$ is the Laplace transform of $e^{at} f(t)$ for $s > s_0 + a$.

Theorem 8.1.6 If f is piecewise continuous on $[0, \infty)$ and of exponential order s_0 , then $\mathcal{L}(f)$ is defined for $s > s_0$.

Section 8.2

Theorem 8.2.1 [Linearity Property] If $F_1, F_2, \dots, F_n$ are Laplace transforms and $c_1, c_2, \dots, c_n$ are constants, then

$$\mathcal{L}^{-1}(c_1 F_1 + c_2 F_2 + \dots + c_n F_n) = c_1 \mathcal{L}^{-1}(F_1) + c_2 \mathcal{L}^{-1}(F_2) + \dots + c_n \mathcal{L}^{-1}(F_n).$$

Theorem 8.2.2 Suppose

$$F(s) = \frac{P(s)}{(s - s_1)(s - s_2) \cdots (s - s_n)}, \quad (8.2.6)$$

where $s_1, s_2, \dots, s_n$ are distinct and P is a polynomial of degree less than n . Then

$$F(s) = \frac{A_1}{s - s_1} + \frac{A_2}{s - s_2} + \dots + \frac{A_n}{s - s_n},$$

where A_i can be computed from (8.2.6) by ignoring the factor $s - s_i$ and setting $s = s_i$ elsewhere.

Section 8.3

Theorem 8.3.1 Suppose f is continuous on $[0, \infty)$ and of exponential order s_0 , and f' is piecewise continuous on $[0, \infty)$. Then f and f' have Laplace transforms for $s > s_0$, and

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0). \quad (8.3.1)$$

Theorem 8.3.2 Suppose f and f' are continuous on $[0, \infty)$ and of exponential order s_0 , and that f'' is piecewise continuous on $[0, \infty)$. Then f, f' , and f'' have Laplace transforms for $s > s_0$,

$$\mathcal{L}(f') = s\mathcal{L}(f) - f(0), \quad (8.3.4)$$

and

$$\mathcal{L}(f'') = s^2 \mathcal{L}(f) - f'(0) - sf(0). \quad (8.3.5)$$

Section 8.4

Theorem 8.4.1 Let g be defined on $[0, \infty)$. Suppose $\tau \geq 0$ and $\mathcal{L}(g(t + \tau))$ exists for $s > s_0$. Then $\mathcal{L}(u(t - \tau)g(t))$ exists for $s > s_0$, and

$$\mathcal{L}(u(t - \tau)g(t)) = e^{-s\tau} \mathcal{L}(g(t + \tau)).$$

Theorem 8.4.2 [Second Shifting Theorem] If $\tau \geq 0$ and $\mathcal{L}(g)$ exists for $s > s_0$ then $\mathcal{L}(u(t - \tau)g(t - \tau))$ exists for $s > s_0$ and

$$\mathcal{L}(u(t - \tau)g(t - \tau)) = e^{-s\tau} \mathcal{L}(g(t)),$$

or, equivalently,

$$\text{if } g(t) \leftrightarrow G(s), \text{ then } u(t - \tau)g(t - \tau) \leftrightarrow e^{-s\tau} G(s). \quad (8.4.12)$$

Section 8.5

Theorem 8.5.1 Suppose a , b , and c are constants ($a \neq 0$), and f is piecewise continuous on $[0, \infty)$ except for jump discontinuities at $t_1, \dots, t_n$, where

$$0 < t_1 < \dots < t_n.$$

Let k_0 and k_1 be arbitrary real numbers. Then there is a unique function y defined on $[0, \infty)$ with these properties:

(a) $y(0) = k_0$ and $y'(0) = k_1$.

(b) y and y' are continuous on $[0, \infty)$.

(c) y'' is defined on every open subinterval of $[0, \infty)$ that does not contain any of the points $t_1, \dots, t_n$, and

$$ay'' + by' + cy = f(t)$$

on every such subinterval.

(d) y'' has limits from the right and left at $t_1, \dots, t_n$.

Section 8.6

Theorem 8.6.2 [The Convolution Theorem] Suppose f and g are piecewise continuous on $[0, \infty)$ and their Laplace transforms converge absolutely at s . If $\mathcal{L}(f) = F$ and $\mathcal{L}(g) = G$, then

$$\mathcal{L}(f * g) = FG.$$

Theorem 8.6.3 Suppose f is continuous on $[0, \infty)$ and has a Laplace transform. Then the solution of the initial value problem

$$ay'' + by' + cy = f(t), \quad y(0) = k_0, \quad y'(0) = k_1, \quad (8.6.13)$$

is

$$y(t) = k_0 y_1(t) + k_1 y_2(t) + \int_0^t w(\tau) f(t - \tau) d\tau, \quad (8.6.14)$$

where y_1 and y_2 satisfy

$$ay_1'' + by_1' + cy_1 = 0, \quad y_1(0) = 1, \quad y_1'(0) = 0, \quad (8.6.15)$$

and

$$ay_2'' + by_2' + cy_2 = 0, \quad y_2(0) = 0, \quad y_2'(0) = 1, \quad (8.6.16)$$

and

$$w(t) = \frac{1}{a} y_2(t). \quad (8.6.17)$$

Section 8.7

Theorem 8.7.1 Suppose $t_0 \geq 0$. For each positive number h , let y_h be the solution of the initial value problem

$$ay_h'' + by_h' + cy_h = f_h(t), \quad y_h(0) = 0, \quad y_h'(0) = 0, \quad (8.7.1)$$

where

$$f_h(t) = \begin{cases} 0, & 0 \leq t < t_0, \\ 1/h, & t_0 \leq t < t_0 + h, \\ 0, & t \geq t_0 + h, \end{cases} \quad (8.7.2)$$

so f_h has unit total impulse equal to the area of the shaded rectangle in Figure 8.7.1. Then

$$\lim_{h \rightarrow 0^+} y_h(t) = u(t - t_0)w(t - t_0), \quad (8.7.3)$$

where

$$w = \mathcal{L}^{-1} \left(\frac{1}{as^2 + bs + c} \right).$$

Section 9.1

Theorem 9.1.1 Suppose $Ly = F$ is normal on (a, b) , let x_0 be a point in (a, b) , and let $k_0, k_1, \dots, k_{n-1}$ be arbitrary real numbers. Then the initial value problem

$$Ly = F, \quad y(x_0) = k_0, \quad y'(x_0) = k_1, \dots, \quad y^{(n-1)}(x_0) = k_{n-1}$$

has a unique solution on (a, b) .

Theorem 9.1.2 If $Ly = 0$ is normal on (a, b) , then a set $\{y_1, y_2, \dots, y_n\}$ of n solutions of $Ly = 0$ on (a, b) is a fundamental set if and only if it's linearly independent on (a, b) .

Theorem 9.1.3 Suppose the homogeneous linear n th order equation

$$P_0(x)y^{(n)} + P_1(x)y^{(n-1)} + \dots + P_n(x)y = 0 \quad (9.1.14)$$

is normal on (a, b) , let $y_1, y_2, \dots, y_n$ be solutions of (9.1.14) on (a, b) , and let x_0 be in (a, b) . Then the Wronskian of $\{y_1, y_2, \dots, y_n\}$ is given by

$$W(x) = W(x_0) \exp \left\{ - \int_{x_0}^x \frac{P_1(t)}{P_0(t)} dt \right\}, \quad a < x < b. \quad (9.1.15)$$

Therefore, either W has no zeros in (a, b) or $W \equiv 0$ on (a, b) .

Theorem 9.1.4 Suppose $Ly = 0$ is normal on (a, b) and let $y_1, y_2, \dots, y_n$ be n solutions of $Ly = 0$ on (a, b) . Then the following statements are equivalent; that is, they are either all true or all false:

- (a) The general solution of $Ly = 0$ on (a, b) is $y = c_1y_1 + c_2y_2 + \dots + c_ny_n$.
- (b) $\{y_1, y_2, \dots, y_n\}$ is a fundamental set of solutions of $Ly = 0$ on (a, b) .
- (c) $\{y_1, y_2, \dots, y_n\}$ is linearly independent on (a, b) .

(d) The Wronskian of $\{y_1, y_2, \dots, y_n\}$ is nonzero at some point in (a, b) .

(e) The Wronskian of $\{y_1, y_2, \dots, y_n\}$ is nonzero at all points in (a, b) .

Theorem 9.1.5 Suppose $Ly = 0$ is normal on (a, b) , c is in (a, b) , and $\alpha_1, \alpha_2, \dots, \alpha_n$ are real numbers, not all zero. If $y_1, y_2, \dots, y_n$ are solutions of $Ly = 0$ such that

$$\alpha_1 y_i(c) + \alpha_2 y_i'(c) + \dots + \alpha_n y_i^{(n-1)}(c) = 0, \quad 1 \leq i \leq n, \quad (9.1.16)$$

Then $\{y_1, y_2, \dots, y_n\}$ isn't linearly independent on (a, b) .

Theorem 9.1.6 Suppose $Ly = F$ is normal on (a, b) . Let y_p be a particular solution of $Ly = F$ on (a, b) , and let $\{y_1, y_2, \dots, y_n\}$ be a fundamental set of solutions of the complementary equation $Ly = 0$ on (a, b) . Then y is a solution of $Ly = F$ on (a, b) if and only if

$$y = y_p + c_1 y_1 + c_2 y_2 + \dots + c_n y_n,$$

where $c_1, c_2, \dots, c_n$ are constants.

Theorem 9.1.7 [The Principle of Superposition] Suppose for each $i = 1, 2, \dots, r$, the function y_{p_i} is a particular solution of $Ly = F_i$ on (a, b) . Then

$$y_p = y_{p_1} + y_{p_2} + \dots + y_{p_r}$$

is a particular solution of

$$Ly = F_1(x) + F_2(x) + \dots + F_r(x)$$

on (a, b) .

Section 9.2

Theorem 9.2.1 If m is a positive integer, then

$$\{e^{ax}, xe^{ax}, \dots, x^{m-1}e^{ax}\} \quad (9.2.15)$$

is a fundamental set of solutions of

$$(D - a)^m y = 0. \quad (9.2.16)$$

Theorem 9.2.2 If $\omega \neq 0$ and m is a positive integer, then

$$\begin{aligned} &\{e^{\lambda x} \cos \omega x, xe^{\lambda x} \cos \omega x, \dots, x^{m-1}e^{\lambda x} \cos \omega x, \\ &\quad e^{\lambda x} \sin \omega x, xe^{\lambda x} \sin \omega x, \dots, x^{m-1}e^{\lambda x} \sin \omega x\} \end{aligned}$$

is a fundamental set of solutions of

$$[(D - \lambda)^2 + \omega^2]^m y = 0.$$

Section 10.2

Theorem 10.2.1 Suppose the coefficient matrix A and the forcing function $\mathbf{f}$ are continuous on (a, b) , let t_0 be in (a, b) , and let $\mathbf{k}$ be an arbitrary constant n -vector. Then the initial value problem

$$\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t), \quad \mathbf{y}(t_0) = \mathbf{k}$$

has a unique solution on (a, b) .

Section 10.3

Theorem 10.3.1 Suppose the $n \times n$ matrix $A = A(t)$ is continuous on (a, b) . Then a set $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ of n solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b) is a fundamental set if and only if it's linearly independent on (a, b) .

Theorem 10.3.2 [Abel's Formula] Suppose the $n \times n$ matrix $A = A(t)$ is continuous on (a, b) , let $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n$ be solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b) , and let t_0 be in (a, b) . Then the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is given by

$$W(t) = W(t_0) \exp \left(\int_{t_0}^t [a_{11}(s) + a_{22}(s) + \dots + a_{nn}(s)] ds \right), \quad a < t < b. \quad (10.3.6)$$

Therefore, either W has no zeros in (a, b) or $W \equiv 0$ on (a, b) .

Theorem 10.3.3 Suppose the $n \times n$ matrix $A = A(t)$ is continuous on (a, b) and let $\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n$ be solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b) . Then the following statements are equivalent; that is, they are either all true or all false:

- (a) The general solution of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b) is $\mathbf{y} = c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \dots + c_n\mathbf{y}_n$, where $c_1, c_2, \dots, c_n$ are arbitrary constants.
- (b) $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is a fundamental set of solutions of $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b) .
- (c) $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is linearly independent on (a, b) .
- (d) The Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is nonzero at some point in (a, b) .
- (e) The Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ is nonzero at all points in (a, b) .

Section 10.4

Theorem 10.4.1 Suppose the $n \times n$ constant matrix A has n real eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ (which need not be distinct) with associated linearly independent eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$. Then the functions

$$\mathbf{y}_1 = \mathbf{x}_1 e^{\lambda_1 t}, \mathbf{y}_2 = \mathbf{x}_2 e^{\lambda_2 t}, \dots, \mathbf{y}_n = \mathbf{x}_n e^{\lambda_n t}$$

form a fundamental set of solutions of $\mathbf{y}' = A\mathbf{y}$; that is, the general solution of this system is

$$\mathbf{y} = c_1\mathbf{x}_1 e^{\lambda_1 t} + c_2\mathbf{x}_2 e^{\lambda_2 t} + \dots + c_n\mathbf{x}_n e^{\lambda_n t}.$$

Section 10.5

Theorem 10.5.1 Suppose the $n \times n$ matrix A has an eigenvalue λ_1 of multiplicity ≥ 2 and the associated eigenspace has dimension 1; that is, all λ_1 -eigenvectors of A are scalar multiples of an eigenvector $\mathbf{x}$. Then there are infinitely many vectors $\mathbf{u}$ such that

$$(A - \lambda_1 I)\mathbf{u} = \mathbf{x}. \quad (10.5.2)$$

Moreover, if $\mathbf{u}$ is any such vector then

$$\mathbf{y}_1 = \mathbf{x}e^{\lambda_1 t} \quad \text{and} \quad \mathbf{y}_2 = \mathbf{u}e^{\lambda_1 t} + \mathbf{x}te^{\lambda_1 t} \quad (10.5.3)$$

are linearly independent solutions of $\mathbf{y}' = A\mathbf{y}$.

Theorem 10.5.2 Suppose the $n \times n$ matrix A has an eigenvalue λ_1 of multiplicity ≥ 3 and the associated eigenspace is one-dimensional; that is, all eigenvectors associated with λ_1 are scalar multiples of the eigenvector $\mathbf{x}$. Then there are infinitely many vectors $\mathbf{u}$ such that

$$(A - \lambda_1 I)\mathbf{u} = \mathbf{x}, \quad (10.5.9)$$

and, if $\mathbf{u}$ is any such vector, there are infinitely many vectors $\mathbf{v}$ such that

$$(A - \lambda_1 I)\mathbf{v} = \mathbf{u}. \quad (10.5.10)$$

If $\mathbf{u}$ satisfies (10.5.9) and $\mathbf{v}$ satisfies (10.5.10), then

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{x}e^{\lambda_1 t}, \\ \mathbf{y}_2 &= \mathbf{u}e^{\lambda_1 t} + \mathbf{x}te^{\lambda_1 t}, \text{ and} \\ \mathbf{y}_3 &= \mathbf{v}e^{\lambda_1 t} + \mathbf{u}te^{\lambda_1 t} + \mathbf{x}\frac{t^2 e^{\lambda_1 t}}{2} \end{aligned}$$

are linearly independent solutions of $\mathbf{y}' = A\mathbf{y}$.

Theorem 10.5.3 Suppose the $n \times n$ matrix A has an eigenvalue λ_1 of multiplicity ≥ 3 and the associated eigenspace is two-dimensional; that is, all eigenvectors of A associated with λ_1 are linear combinations of two linearly independent eigenvectors $\mathbf{x}_1$ and $\mathbf{x}_2$. Then there are constants α and β (not both zero) such that if

$$\mathbf{x}_3 = \alpha\mathbf{x}_1 + \beta\mathbf{x}_2, \quad (10.5.12)$$

then there are infinitely many vectors $\mathbf{u}$ such that

$$(A - \lambda_1 I)\mathbf{u} = \mathbf{x}_3. \quad (10.5.13)$$

If $\mathbf{u}$ satisfies (10.5.13), then

$$\begin{aligned} \mathbf{y}_1 &= \mathbf{x}_1 e^{\lambda_1 t}, \\ \mathbf{y}_2 &= \mathbf{x}_2 e^{\lambda_1 t}, \text{ and} \\ \mathbf{y}_3 &= \mathbf{u}e^{\lambda_1 t} + \mathbf{x}_3 t e^{\lambda_1 t}, \end{aligned} \quad (10.5.14)$$

are linearly independent solutions of $\mathbf{y}' = A\mathbf{y}$.

Section 10.6

Theorem 10.6.1 Let A be an $n \times n$ matrix with real entries. Let $\lambda = \alpha + i\beta$ ($\beta \neq 0$) be a complex eigenvalue of A and let $\mathbf{x} = \mathbf{u} + i\mathbf{v}$ be an associated eigenvector, where $\mathbf{u}$ and $\mathbf{v}$ have real components. Then $\mathbf{u}$ and $\mathbf{v}$ are both nonzero and

$$\mathbf{y}_1 = e^{\alpha t}(\mathbf{u} \cos \beta t - \mathbf{v} \sin \beta t) \quad \text{and} \quad \mathbf{y}_2 = e^{\alpha t}(\mathbf{u} \sin \beta t + \mathbf{v} \cos \beta t),$$

which are the real and imaginary parts of

$$e^{\alpha t}(\cos \beta t + i \sin \beta t)(\mathbf{u} + i\mathbf{v}), \quad (10.6.3)$$

are linearly independent solutions of $\mathbf{y}' = A\mathbf{y}$.

Section 10.7

Theorem 10.7.1 Suppose the $n \times n$ matrix function A and the n -vector function $\mathbf{f}$ are continuous on (a, b) . Let $\mathbf{y}_p$ be a particular solution of $\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t)$ on (a, b) , and let $\{\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n\}$ be a fundamental set of solutions of the complementary equation $\mathbf{y}' = A(t)\mathbf{y}$ on (a, b) . Then $\mathbf{y}$ is a solution of $\mathbf{y}' = A(t)\mathbf{y} + \mathbf{f}(t)$ on (a, b) if and only if

$$\mathbf{y} = \mathbf{y}_p + c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \dots + c_n\mathbf{y}_n,$$

where $c_1, c_2, \dots, c_n$ are constants.

Section 11.1

Theorem 11.1.1 Problems 1–5 have no negative eigenvalues. Moreover, $\lambda = 0$ is an eigenvalue of Problems 2 and 5, with associated eigenfunction $y_0 = 1$, but $\lambda = 0$ isn't an eigenvalue of Problems 1, 3, or 4.

Theorem 11.1.2 The eigenvalue problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0$$

has infinitely many positive eigenvalues $\lambda_n = n^2\pi^2/L^2$, with associated eigenfunctions

$$y_n = \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \dots$$

There are no other eigenvalues.

Theorem 11.1.3 The eigenvalue problem

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y'(L) = 0$$

has the eigenvalue $\lambda_0 = 0$, with associated eigenfunction $y_0 = 1$, and infinitely many positive eigenvalues $\lambda_n = n^2\pi^2/L^2$, with associated eigenfunctions

$$y_n = \cos \frac{n\pi x}{L}, \quad n = 1, 2, 3, \dots$$

There are no other eigenvalues.

Theorem 11.1.4 *The eigenvalue problem*

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(L) = 0$$

has infinitely many positive eigenvalues $\lambda_n = (2n - 1)^2 \pi^2 / 4L^2$, with associated eigenfunctions

$$y_n = \sin \frac{(2n - 1)\pi x}{2L}, \quad n = 1, 2, 3, \dots$$

There are no other eigenvalues.

Theorem 11.1.5 *The eigenvalue problem*

$$y'' + \lambda y = 0, \quad y'(0) = 0, \quad y(L) = 0$$

has infinitely many positive eigenvalues $\lambda_n = (2n - 1)^2 \pi^2 / 4L^2$, with associated eigenfunctions

$$y_n = \cos \frac{(2n - 1)\pi x}{2L}, \quad n = 1, 2, 3, \dots$$

There are no other eigenvalues.

Theorem 11.1.6 *The eigenvalue problem*

$$y'' + \lambda y = 0, \quad y(-L) = y(L), \quad y'(-L) = y'(L),$$

has the eigenvalue $\lambda_0 = 0$, with associated eigenfunction $y_0 = 1$ and infinitely many positive eigenvalues $\lambda_n = n^2 \pi^2 / L^2$, with associated eigenfunctions

$$y_{1n} = \cos \frac{n\pi x}{L} \quad \text{and} \quad y_{2n} = \sin \frac{n\pi x}{L}, \quad n = 1, 2, 3, \dots$$

There are no other eigenvalues.

Section 11.2

Theorem 11.2.1 Suppose the functions $\phi_1, \phi_2, \phi_3, \dots$, are orthogonal on $[a, b]$ and

$$\int_a^b \phi_n^2(x) dx \neq 0, \quad n = 1, 2, 3, \dots \quad (11.2.1)$$

Let $c_1, c_2, c_3, \dots$ be constants such that the partial sums $f_N(x) = \sum_{m=1}^N c_m \phi_m(x)$ satisfy the inequalities

$$|f_N(x)| \leq M, \quad a \leq x \leq b, \quad N = 1, 2, 3, \dots$$

for some constant $M < \infty$. Suppose also that the series

$$f(x) = \sum_{m=1}^{\infty} c_m \phi_m(x) \quad (11.2.2)$$

converges and is integrable on $[a, b]$. Then

$$c_n = \frac{\int_a^b f(x) \phi_n(x) dx}{\int_a^b \phi_n^2(x) dx}, \quad n = 1, 2, 3, \dots \quad (11.2.3)$$

Theorem 11.2.4 *If f is piecewise smooth on $[-L, L]$, then the Fourier series*

$$F(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \quad (11.2.8)$$

of f on $[-L, L]$ converges for all x in $[-L, L]$; moreover,

$$F(x) = \begin{cases} f(x) & \text{if } -L < x < L \text{ and } f \text{ is continuous at } x \\ \frac{f(x-) + f(x+)}{2} & \text{if } -L < x < L \text{ and } f \text{ is discontinuous at } x \\ \frac{f(-L+) + f(L-)}{2} & \text{if } x = L \text{ or } x = -L. \end{cases}$$

Theorem 11.2.5 *Suppose u is even and v is odd on $[-L, L]$. Then:*

$$\begin{aligned} \text{(a)} \quad & \int_{-L}^L u(x) dx = 2 \int_0^L u(x) dx, & \text{(b)} \quad & \int_{-L}^L v(x) dx = 0, \\ \text{(c)} \quad & \int_{-L}^L u(x) \cos \frac{n\pi x}{L} dx = 2 \int_0^L u(x) \cos \frac{n\pi x}{L} dx, \\ \text{(d)} \quad & \int_{-L}^L v(x) \sin \frac{n\pi x}{L} dx = 2 \int_0^L v(x) \sin \frac{n\pi x}{L} dx, \\ \text{(e)} \quad & \int_{-L}^L u(x) \sin \frac{n\pi x}{L} dx = 0 \quad \text{and} \quad \text{(f)} \quad \int_{-L}^L v(x) \cos \frac{n\pi x}{L} dx = 0. \end{aligned}$$

Theorem 11.2.6 *Suppose f is integrable on $[-L, L]$.*

(a) If f is even, the Fourier series of f on $[-L, L]$ is

$$F(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L},$$

where

$$a_0 = \frac{1}{L} \int_0^L f(x) dx \quad \text{and} \quad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx, \quad n \geq 1.$$

(b) If f is odd, the Fourier series of f on $[-L, L]$ is

$$F(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L},$$

where

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx.$$

Section 11.3

Theorem 11.3.1 *If f is piecewise smooth on $[0, L]$, then the Fourier cosine series*

$$C(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}$$

of f on $[0, L]$, with

$$a_0 = \frac{1}{L} \int_0^L f(x) dx \quad \text{and} \quad a_n = \frac{2}{L} \int_0^L f(x) \cos \frac{n\pi x}{L} dx, \quad n = 1, 2, 3, \dots,$$

converges for all x in $[0, L]$; moreover,

$$C(x) = \begin{cases} f(0+) & \text{if } x = 0 \\ f(x) & \text{if } 0 < x < L \text{ and } f \text{ is continuous at } x \\ \frac{f(x-) + f(x+)}{2} & \text{if } 0 < x < L \text{ and } f \text{ is discontinuous at } x \\ f(L-) & \text{if } x = L. \end{cases}$$

Theorem 11.3.2 *If f is piecewise smooth on $[0, L]$, then the Fourier sine series*

$$S(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$

of f on $[0, L]$, with

$$b_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx,$$

converges for all x in $[0, L]$; moreover,

$$S(x) = \begin{cases} 0 & \text{if } x = 0 \\ f(x) & \text{if } 0 < x < L \text{ and } f \text{ is continuous at } x \\ \frac{f(x-) + f(x+)}{2} & \text{if } 0 < x < L \text{ and } f \text{ is discontinuous at } x \\ 0 & \text{if } x = L. \end{cases}$$

Theorem 11.3.3 *If f is piecewise smooth on $[0, L]$, then the mixed Fourier cosine series*

$$C_M(x) = \sum_{n=1}^{\infty} c_n \cos \frac{(2n-1)\pi x}{2L}$$

of f on $[0, L]$, with

$$c_n = \frac{2}{L} \int_0^L f(x) \cos \frac{(2n-1)\pi x}{2L} dx,$$

converges for all x in $[0, L]$; moreover,

$$C_M(x) = \begin{cases} f(0+) & \text{if } x = 0 \\ f(x) & \text{if } 0 < x < L \text{ and } f \text{ is continuous at } x \\ \frac{f(x-) + f(x+)}{2} & \text{if } 0 < x < L \text{ and } f \text{ is discontinuous at } x \\ 0 & \text{if } x = L. \end{cases}$$

Theorem 11.3.4 *If f is piecewise smooth on $[0, L]$, then the mixed Fourier sine series*

$$S_M(x) = \sum_{n=1}^{\infty} d_n \sin \frac{(2n-1)\pi x}{2L}$$

of f on $[0, L]$, with

$$d_n = \frac{2}{L} \int_0^L f(x) \sin \frac{(2n-1)\pi x}{2L} dx,$$

converges for all x in $[0, L]$; moreover,

$$S_M(x) = \begin{cases} 0 & \text{if } x = 0 \\ f(x) & \text{if } 0 < x < L \text{ and } f \text{ is continuous at } x \\ \frac{f(x-) + f(x+)}{2} & \text{if } 0 < x < L \text{ and } f \text{ is discontinuous at } x \\ f(L-) & \text{if } x = L. \end{cases}$$

Theorem 11.3.5

(a) *If $f'(0) = f'(L) = 0$, f'' is continuous, and f''' is piecewise continuous on $[0, L]$, then*

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{L}, \quad 0 \leq x \leq L, \quad (11.3.3)$$

with

$$a_0 = \frac{1}{L} \int_0^L f(x) dx \quad \text{and} \quad a_n = \frac{2L^2}{n^3\pi^3} \int_0^L f'''(x) \sin \frac{n\pi x}{L} dx, \quad n \geq 1. \quad (11.3.4)$$

Now suppose f' is continuous and f'' is piecewise continuous on $[0, L]$.

(b) *If $f(0) = f(L) = 0$, then*

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}, \quad 0 \leq x \leq L,$$

with

$$b_n = -\frac{2L}{n^2\pi^2} \int_0^L f''(x) \sin \frac{n\pi x}{L} dx. \quad (11.3.5)$$

(c) *If $f'(0) = f'(L) = 0$, then*

$$f(x) = \sum_{n=1}^{\infty} c_n \cos \frac{(2n-1)\pi x}{2L}, \quad 0 \leq x \leq L,$$

with

$$c_n = -\frac{8L}{(2n-1)^2\pi^2} \int_0^L f''(x) \cos \frac{(2n-1)\pi x}{2L} dx. \quad (11.3.6)$$

(d) *If $f(0) = f'(L) = 0$, then*

$$f(x) = \sum_{n=1}^{\infty} d_n \sin \frac{(2n-1)\pi x}{2L}, \quad 0 \leq x \leq L,$$

with

$$d_n = -\frac{8L}{(2n-1)^2\pi^2} \int_0^L f''(x) \sin \frac{(2n-1)\pi x}{2L} dx. \quad (11.3.7)$$

Section 12.1

Theorem 12.1.2 *A convergent infinite series*

$$W(z) = \sum_{n=1}^{\infty} w_n(z)$$

can be differentiated term by term on a closed interval $[z_1, z_2]$ to obtain

$$W'(z) = \sum_{n=1}^{\infty} w'_n(z)$$

(where the derivatives at $z = z_1$ and $z = z_2$ are one-sided) provided that w'_n is continuous on $[z_1, z_2]$ and

$$|w'_n(z)| \leq M_n, \quad z_1 \leq z \leq z_2, \quad n = 1, 2, 3, \dots,$$

where $M_1, M_2, \dots, M_n, \dots$, are constants such that the series $\sum_{n=1}^{\infty} M_n$ converges.

Section 12.2

Theorem 12.2.2 *If f and g are piecewise smooth on $[0, L]$, then the series in (12.2.8) converges for all (x, t) and can be written as*

$$u(x, t) = \frac{1}{2} [S_f(x + at) + S_f(x - at)] + \frac{1}{2a} \int_{x-at}^{x+at} S_g(\tau) d\tau. \quad (12.2.9)$$

Theorem 12.2.3 *Suppose h is differentiable on $[0, L]$; that is, $h'(x)$ exists for $0 < x < L$, and the one-sided derivatives*

$$h'_+(0) = \lim_{x \rightarrow 0+} \frac{h(x) - h(0)}{x} \quad \text{and} \quad h'_-(L) = \lim_{x \rightarrow L-} \frac{h(x) - h(L)}{x - L}$$

both exist.

(a) *Let p be the odd periodic extension of h to $(-\infty, \infty)$; that is,*

$$p(x) = \begin{cases} h(x), & 0 \leq x \leq L, \\ -h(-x), & -L < x < 0, \end{cases} \quad \text{and } p(x + 2L) = p(x), \quad -\infty < x < \infty.$$

Then p is differentiable on $(-\infty, \infty)$ if and only if

$$h(0) = h(L) = 0. \quad (12.2.13)$$

(b) *Let q be the even periodic extension of h to $(-\infty, \infty)$; that is,*

$$q(x) = \begin{cases} h(x), & 0 \leq x \leq L, \\ h(-x), & -L < x < 0, \end{cases} \quad \text{and } q(x + 2L) = q(x), \quad -\infty < x < \infty.$$

Then q is differentiable on $(-\infty, \infty)$ if and only if

$$h'_+(0) = h'_-(L) = 0. \quad (12.2.14)$$

Theorem 12.2.4 *The formal solution of (12.2.1) is an actual solution if g is differentiable on $[0, L]$ and*

$$g(0) = g(L) = 0, \quad (12.2.15)$$

while f is twice differentiable on $[0, L]$ and

$$f(0) = f(L) = 0 \quad (12.2.16)$$

and

$$f''_+(0) = f''_-(L) = 0. \quad (12.2.17)$$

Section 13.1

Theorem 13.1.1 *If z_1 and z_2 are solutions of $Ly = 0$ such that either $B_1(z_1) = B_1(z_2) = 0$ or $B_2(z_1) = B_2(z_2) = 0$, then $\{z_1, z_2\}$ is linearly dependent. Equivalently, if $\{z_1, z_2\}$ is linearly independent, then*

$$B_1^2(z_1) + B_1^2(z_2) \neq 0 \quad \text{and} \quad B_2^2(z_1) + B_2^2(z_2) \neq 0.$$

Theorem 13.1.2 *The following statements are equivalent; that is, they are either all true or all false.*

(a) *There's a fundamental set $\{z_1, z_2\}$ of solutions of $Ly = 0$ such that*

$$B_1(z_1)B_2(z_2) - B_1(z_2)B_2(z_1) \neq 0. \quad (13.1.8)$$

(b) *If $\{y_1, y_2\}$ is a fundamental set of solutions of $Ly = 0$ then*

$$B_1(y_1)B_2(y_2) - B_1(y_2)B_2(y_1) \neq 0. \quad (13.1.9)$$

(c) *For each continuous F and pair of constants (k_1, k_2) , the boundary value problem*

$$Ly = F, \quad B_1(y) = k_1, \quad B_2(y) = k_2$$

has a unique solution.

(d) *The homogeneous boundary value problem*

$$Ly = 0, \quad B_1(y) = 0, \quad B_2(y) = 0 \quad (13.1.10)$$

has only the trivial solution $y = 0$.

(e) *The homogeneous equation $Ly = 0$ has linearly independent solutions z_1 and z_2 such that $B_1(z_1) = 0$ and $B_2(z_2) = 0$.*

Theorem 13.1.3 *Suppose the homogeneous boundary value problem*

$$Ly = 0, \quad B_1(y) = 0, \quad B_2(y) = 0 \quad (13.1.15)$$

has only the trivial solution. Let y_1 and y_2 be linearly independent solutions of $Ly = 0$ such that $B_1(y_1) = 0$ and $B_2(y_2) = 0$, and let

$$W = y_1 y_2' - y_1' y_2.$$

Then the unique solution of

$$Ly = F, \quad B_1(y) = 0, \quad B_2(y) = 0 \quad (13.1.16)$$

is

$$y(x) = y_1(x) \int_x^b \frac{F(t)y_2(t)}{P_0(t)W(t)} dt + y_2(x) \int_a^x \frac{F(t)y_1(t)}{P_0(t)W(t)} dt. \quad (13.1.17)$$

Theorem 13.1.4 Suppose the homogeneous boundary value problem

$$Ly = 0, \quad B_1(y) = 0, \quad B_2(y) = 0 \quad (13.1.22)$$

has a nontrivial solution y_1 , and let y_2 be any solution of $Ly = 0$ that isn't a constant multiple of y_1 . Let $W = y_1 y_2' - y_1' y_2$. If

$$\int_a^b \frac{F(t)y_1(t)}{P_0(t)W(t)} dt = 0, \quad (13.1.23)$$

then the nonhomogeneous boundary value problem

$$Ly = F, \quad B_1(y) = 0, \quad B_2(y) = 0 \quad (13.1.24)$$

has infinitely many solutions, all of the form $y = y_p + c_1 y_1$, where

$$y_p = y_1(x) \int_x^b \frac{F(t)y_2(t)}{P_0(t)W(t)} dt + y_2(x) \int_a^x \frac{F(t)y_1(t)}{P_0(t)W(t)} dt$$

and c_1 is a constant. If

$$\int_a^b \frac{F(t)y_1(t)}{P_0(t)W(t)} dt \neq 0,$$

then (13.1.24) has no solution.

Section 13.2

Theorem 13.2.1 If P_0 , P_1 , P_2 , and R are continuous and P_0 and R are positive on a closed interval $[a, b]$, then the equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y + \lambda R(x)y = 0 \quad (13.2.5)$$

can be rewritten as

$$(p(x)y')' + q(x)y + \lambda r(x)y = 0, \quad (13.2.6)$$

where p , p' , q and r are continuous and p and r are positive on $[a, b]$.

Theorem 13.2.2 If

$$Ly = (p(x)y')' + q(x)y$$

and u and v are twice continuously differentiable functions on $[a, b]$ that satisfy the boundary conditions $B_1(y) = 0$ and $B_2(y) = 0$, then

$$\int_a^b [u(x)Lv(x) - v(x)Lu(x)] dx = 0. \quad (13.2.21)$$

Theorem 13.2.3 If $\lambda = p + qi$ with $q \neq 0$ then the boundary value problem

$$Ly + \lambda r(x)y = 0, \quad B_1(y) = 0, \quad B_2(y) = 0$$

has only the trivial solution.

Theorem 13.2.4 *If λ_1 and λ_2 are distinct eigenvalues of the Sturm–Liouville problem*

$$Ly + \lambda r(x)y = 0, \quad B_1(y) = 0, \quad B_2(y) = 0 \quad (13.2.25)$$

with associated eigenfunctions u and v respectively, then

$$\int_a^b r(x)u(x)v(x) dx = 0. \quad (13.2.26)$$

Theorem 13.2.5 *If $u \not\equiv 0$ and v both satisfy*

$$Ly + \lambda r(x)y = 0, \quad B_1(y) = 0, \quad B_2(y) = 0,$$

then $v = cu$ for some constant c .

Theorem 13.2.6 *The set of all eigenvalues of the Sturm–Liouville problem*

$$Ly + \lambda r(x)y = 0, \quad B_1(y) = 0, \quad B_2(y) = 0$$

can be ordered as

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots,$$

and

$$\lim_{n \rightarrow \infty} \lambda_n = \infty.$$

For each n , if y_n is an arbitrary λ_n -eigenfunction, then every λ_n -eigenfunction is a constant multiple of y_n . If $m \neq n$, y_m and y_n are orthogonal on $[a, b]$ with respect to $r = r(x)$; that is,

$$\int_a^b r(x)y_m(x)y_n(x) dx = 0. \quad (13.2.27)$$

Theorem 13.2.7 *Let $\lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots$ be the eigenvalues of the Sturm–Liouville problem*

$$Ly + \lambda r(x)y = 0, \quad B_1(y) = 0, \quad B_2(y) = 0,$$

with associated eigenfunctions $y_1, y_2, \dots, y_n, \dots$. Suppose f is piecewise smooth (Definition 11.2.3) on $[a, b]$. For each n , let

$$c_n = \frac{\int_a^b r(x)f(x)y_n(x) dx}{\int_a^b r(x)y_n^2(x) dx}.$$

Then

$$\frac{f(x-) + f(x+)}{2} = \sum_{n=1}^{\infty} c_n y_n(x)$$

for all x in the open interval (a, b) .