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Section 1.2

Example 1.2.1 If a is any positive constant, the circle

$$x^2 + y^2 = a^2 \tag{1.2.3}$$

is an integral curve of

$$y' = -\frac{x}{y}. \tag{1.2.4}$$

To see this, note that the only functions whose graphs are segments of (1.2.3) are

$$y_1 = \sqrt{a^2 - x^2} \quad \text{and} \quad y_2 = -\sqrt{a^2 - x^2}.$$

We leave it to you to verify that these functions both satisfy (1.2.4) on the open interval $(-a, a)$. However, (1.2.3) is not a solution curve of (1.2.4), since it's not the graph of a function.

Example 1.2.2 Verify that

$$y = \frac{x^2}{3} + \frac{1}{x} \quad (1.2.5)$$

is a solution of

$$xy' + y = x^2 \quad (1.2.6)$$

on $(0, \infty)$ and on $(-\infty, 0)$.

Solution Substituting (1.2.5) and

$$y' = \frac{2x}{3} - \frac{1}{x^2}$$

into (1.2.6) yields

$$xy'(x) + y(x) = x \left(\frac{2x}{3} - \frac{1}{x^2} \right) + \left(\frac{x^2}{3} + \frac{1}{x} \right) = x^2$$

for all $x \neq 0$. Therefore y is a solution of (1.2.6) on $(-\infty, 0)$ and $(0, \infty)$. However, y isn't a solution of the differential equation on any open interval that contains $x = 0$, since y is not defined at $x = 0$.

Figure 1.2.1 shows the graph of (1.2.5). The part of the graph of (1.2.5) on $(0, \infty)$ is a solution curve of (1.2.6), as is the part of the graph on $(-\infty, 0)$.

Example 1.2.3 Show that if c_1 and c_2 are constants then

$$y = (c_1 + c_2x)e^{-x} + 2x - 4 \quad (1.2.7)$$

is a solution of

$$y'' + 2y' + y = 2x \quad (1.2.8)$$

on $(-\infty, \infty)$.

Solution Differentiating (1.2.7) twice yields

$$y' = -(c_1 + c_2x)e^{-x} + c_2e^{-x} + 2$$

and

$$y'' = (c_1 + c_2x)e^{-x} - 2c_2e^{-x},$$

so

$$\begin{aligned} y'' + 2y' + y &= (c_1 + c_2x)e^{-x} - 2c_2e^{-x} \\ &\quad + 2[-(c_1 + c_2x)e^{-x} + c_2e^{-x} + 2] \\ &\quad + (c_1 + c_2x)e^{-x} + 2x - 4 \\ &= (1 - 2 + 1)(c_1 + c_2x)e^{-x} + (-2 + 2)c_2e^{-x} \\ &\quad + 4 + 2x - 4 = 2x \end{aligned}$$

for all values of x . Therefore y is a solution of (1.2.8) on $(-\infty, \infty)$.

Example 1.2.4 Find all solutions of

$$y^{(n)} = e^{2x}. \quad (1.2.9)$$

Solution Integrating (1.2.9) yields

$$y^{(n-1)} = \frac{e^{2x}}{2} + k_1,$$

where k_1 is a constant. If $n \geq 2$, integrating again yields

$$y^{(n-2)} = \frac{e^{2x}}{4} + k_1x + k_2.$$

If $n \geq 3$, repeatedly integrating yields

$$y = \frac{e^{2x}}{2^n} + k_1 \frac{x^{n-1}}{(n-1)!} + k_2 \frac{x^{n-2}}{(n-2)!} + \cdots + k_n, \quad (1.2.10)$$

where $k_1, k_2, \dots, k_n$ are constants. This shows that every solution of (1.2.9) has the form (1.2.10) for some choice of the constants $k_1, k_2, \dots, k_n$. On the other hand, differentiating (1.2.10) n times shows that if $k_1, k_2, \dots, k_n$ are arbitrary constants, then the function y in (1.2.10) satisfies (1.2.9).

Since the constants $k_1, k_2, \dots, k_n$ in (1.2.10) are arbitrary, so are the constants

$$\frac{k_1}{(n-1)!}, \frac{k_2}{(n-2)!}, \dots, k_n.$$

Therefore Example 1.2.4 actually shows that all solutions of (1.2.9) can be written as

$$y = \frac{e^{2x}}{2^n} + c_1 + c_2x + \cdots + c_nx^{n-1},$$

where we renamed the arbitrary constants in (1.2.10) to obtain a simpler formula. As a general rule, arbitrary constants appearing in solutions of differential equations should be simplified if possible. You'll see examples of this throughout the text.

Example 1.2.5 Find a solution of

$$y' = x^3$$

such that $y(1) = 2$.

Solution At the beginning of this section we saw that the solutions of $y' = x^3$ are

$$y = \frac{x^4}{4} + c.$$

To determine a value of c such that $y(1) = 2$, we set $x = 1$ and $y = 2$ here to obtain

$$2 = y(1) = \frac{1}{4} + c, \quad \text{so} \quad c = \frac{7}{4}.$$

Therefore the required solution is

$$y = \frac{x^4 + 7}{4}.$$

Figure 1.2.2 shows the graph of this solution. Note that imposing the condition $y(1) = 2$ is equivalent to requiring the graph of y to pass through the point $(1, 2)$.

Example 1.2.6 In Example 1.2.5 we saw that

$$y = \frac{x^4 + 7}{4} \quad (1.2.13)$$

is a solution of the initial value problem

$$y' = x^3, \quad y(1) = 2.$$

Since the function in (1.2.13) is defined for all x , the interval of validity of this solution is $(-\infty, \infty)$.

Example 1.2.7 In Example 1.2.2 we verified that

$$y = \frac{x^2}{3} + \frac{1}{x} \quad (1.2.14)$$

is a solution of

$$xy' + y = x^2$$

on $(0, \infty)$ and on $(-\infty, 0)$. By evaluating (1.2.14) at $x = \pm 1$, you can see that (1.2.14) is a solution of the initial value problems

$$xy' + y = x^2, \quad y(1) = \frac{4}{3} \quad (1.2.15)$$

and

$$xy' + y = x^2, \quad y(-1) = -\frac{2}{3}. \quad (1.2.16)$$

The interval of validity of (1.2.14) as a solution of (1.2.15) is $(0, \infty)$, since this is the largest interval that contains $x_0 = 1$ on which (1.2.14) is defined. Similarly, the interval of validity of (1.2.14) as a solution of (1.2.16) is $(-\infty, 0)$, since this is the largest interval that contains $x_0 = -1$ on which (1.2.14) is defined.

Example 1.2.8 An object falls under the influence of gravity near Earth's surface, where it can be assumed that the magnitude of the acceleration due to gravity is a constant g .

- (a) Construct a mathematical model for the motion of the object in the form of an initial value problem for a second order differential equation, assuming that the altitude and velocity of the object at time $t = 0$ are known. Assume that gravity is the only force acting on the object.
- (b) Solve the initial value problem derived in (a) to obtain the altitude as a function of time.

Solution (a) Let $y(t)$ be the altitude of the object at time t . Since the acceleration of the object has constant magnitude g and is in the downward (negative) direction, y satisfies the second order equation

$$y'' = -g,$$

where the prime now indicates differentiation with respect to t . If y_0 and v_0 denote the altitude and velocity when $t = 0$, then y is a solution of the initial value problem

$$y'' = -g, \quad y(0) = y_0, \quad y'(0) = v_0. \quad (1.2.17)$$

Solution (b) Integrating (1.2.17) twice yields

$$\begin{aligned}y' &= -gt + c_1, \\y &= -\frac{gt^2}{2} + c_1t + c_2.\end{aligned}$$

Imposing the initial conditions $y(0) = y_0$ and $y'(0) = v_0$ in these two equations shows that $c_1 = v_0$ and $c_2 = y_0$. Therefore the solution of the initial value problem (1.2.17) is

$$y = -\frac{gt^2}{2} + v_0t + y_0.$$

Section 2.1

Example 2.1.1 The first order equations

$$\begin{aligned}x^2y' + 3y &= x^2, \\xy' - 8x^2y &= \sin x, \\xy' + (\ln x)y &= 0, \\y' &= x^2y - 2,\end{aligned}$$

are not in the form (2.1.1), but they are linear, since they can be rewritten as

$$\begin{aligned}y' + \frac{3}{x^2}y &= 1, \\y' - 8xy &= \frac{\sin x}{x}, \\y' + \frac{\ln x}{x}y &= 0, \\y' - x^2y &= -2.\end{aligned}$$

Example 2.1.2 Here are some nonlinear first order equations:

$$\begin{aligned}xy' + 3y^2 &= 2x && \text{(because } y \text{ is squared),} \\yy' &= 3 && \text{(because of the product } yy'), \\y' + xe^y &= 12 && \text{(because of } e^y).\end{aligned}$$

Example 2.1.3 Let a be a constant.

(a) Find the general solution of

$$y' - ay = 0. \tag{2.1.6}$$

(b) Solve the initial value problem

$$y' - ay = 0, \quad y(x_0) = y_0.$$

Solution (a) You already know from calculus that if c is any constant, then $y = ce^{ax}$ satisfies (2.1.6). However, let's pretend you've forgotten this, and use this problem to illustrate a general method for solving a homogeneous linear first order equation.

We know that (2.1.6) has the trivial solution $y \equiv 0$. Now suppose y is a nontrivial solution of (2.1.6). Then, since a differentiable function must be continuous, there must be some open interval I on which y has no zeros. We rewrite (2.1.6) as

$$\frac{y'}{y} = a$$

for x in I . Integrating this shows that

$$\ln |y| = ax + k, \quad \text{so} \quad |y| = e^k e^{ax},$$

where k is an arbitrary constant. Since e^{ax} can never equal zero, y has no zeros, so y is either always positive or always negative. Therefore we can rewrite y as

$$y = ce^{ax} \tag{2.1.7}$$

where

$$c = \begin{cases} e^k & \text{if } y > 0, \\ -e^k & \text{if } y < 0. \end{cases}$$

This shows that every nontrivial solution of (2.1.6) is of the form $y = ce^{ax}$ for some nonzero constant c . Since setting $c = 0$ yields the trivial solution, *all* solutions of (2.1.6) are of the form (2.1.7). Conversely, (2.1.7) is a solution of (2.1.6) for every choice of c , since differentiating (2.1.7) yields $y' = ace^{ax} = ay$.

Solution (b) Imposing the initial condition $y(x_0) = y_0$ yields $y_0 = ce^{ax_0}$, so $c = y_0 e^{-ax_0}$ and

$$y = y_0 e^{-ax_0} e^{ax} = y_0 e^{a(x-x_0)}.$$

Figure 2.1.1 shows the graphs of this function with $x_0 = 0$, $y_0 = 1$, and various values of a .

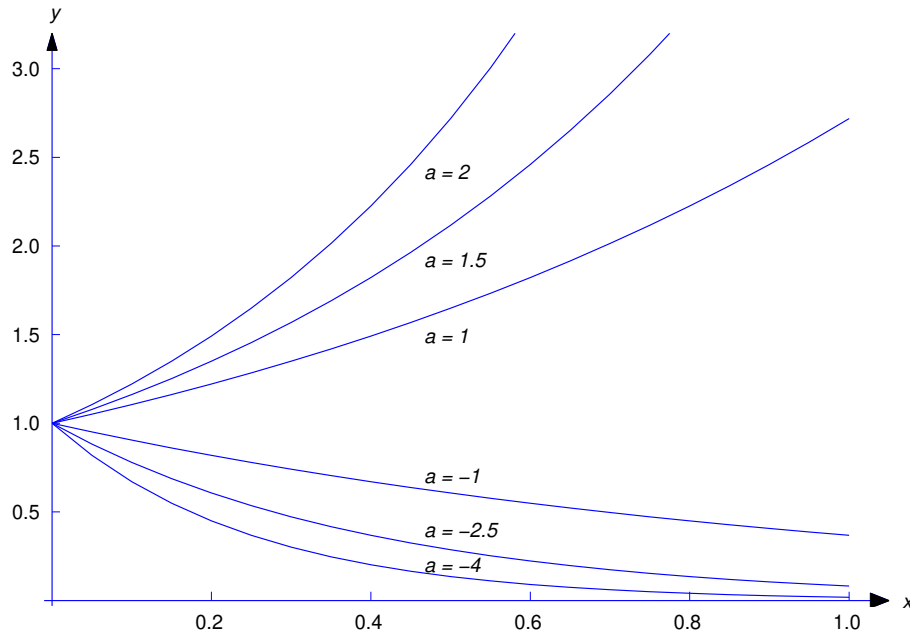


Figure 2.1.1 Solutions of $y' - ay = 0$, $y(0) = 1$

Example 2.1.4 (a) Find the general solution of

$$xy' + y = 0. \tag{2.1.8}$$

(b) Solve the initial value problem

$$xy' + y = 0, \quad y(1) = 3. \quad (2.1.9)$$

Solution (a) We rewrite (2.1.8) as

$$y' + \frac{1}{x}y = 0, \quad (2.1.10)$$

where x is restricted to either $(-\infty, 0)$ or $(0, \infty)$. If y is a nontrivial solution of (2.1.10), there must be some open interval I on which y has no zeros. We can rewrite (2.1.10) as

$$\frac{y'}{y} = -\frac{1}{x}$$

for x in I . Integrating shows that

$$\ln |y| = -\ln |x| + k, \quad \text{so} \quad |y| = \frac{e^k}{|x|}.$$

Since a function that satisfies the last equation can't change sign on either $(-\infty, 0)$ or $(0, \infty)$, we can rewrite this result more simply as

$$y = \frac{c}{x} \quad (2.1.11)$$

where

$$c = \begin{cases} e^k & \text{if } y > 0, \\ -e^k & \text{if } y < 0. \end{cases}$$

We've now shown that every solution of (2.1.10) is given by (2.1.11) for some choice of c . (Even though we assumed that y was nontrivial to derive (2.1.11), we can get the trivial solution by setting $c = 0$ in (2.1.11).) Conversely, any function of the form (2.1.11) is a solution of (2.1.10), since differentiating (2.1.11) yields

$$y' = -\frac{c}{x^2},$$

and substituting this and (2.1.11) into (2.1.10) yields

$$\begin{aligned} y' + \frac{1}{x}y &= -\frac{c}{x^2} + \frac{1}{x} \frac{c}{x} \\ &= -\frac{c}{x^2} + \frac{c}{x^2} = 0. \end{aligned}$$

Figure 2.1.2 shows the graphs of some solutions corresponding to various values of c .

Solution (b) Imposing the initial condition $y(1) = 3$ in (2.1.11) yields $c = 3$. Therefore the solution of (2.1.9) is

$$y = \frac{3}{x}.$$

The interval of validity of this solution is $(0, \infty)$.

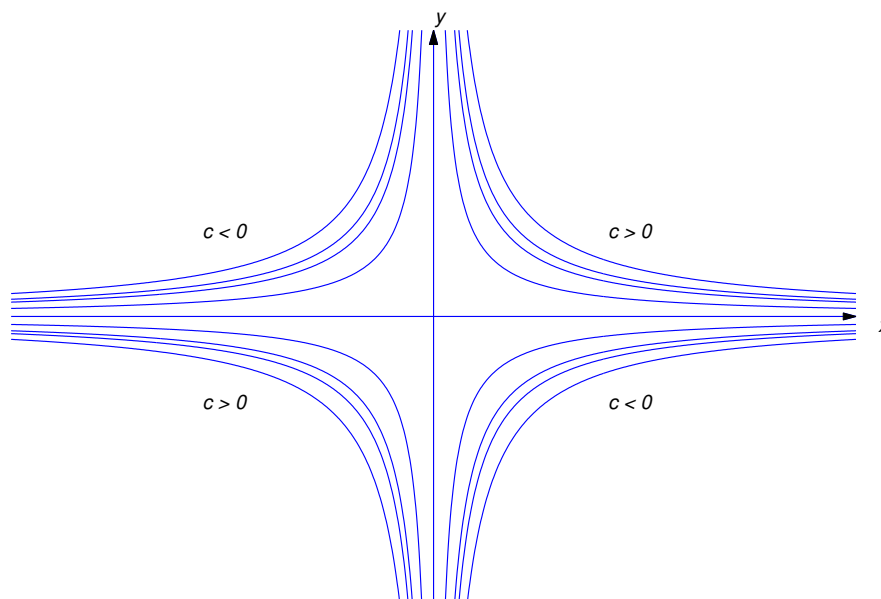


Figure 2.1.2 Solutions of $xy' + y = 0$ on $(0, \infty)$ and $(-\infty, 0)$

Example 2.1.5 Find the general solution of

$$y' + 2y = x^3 e^{-2x}. \quad (2.1.18)$$

Solution By applying (a) of Example 2.1.3 with $a = -2$, we see that $y_1 = e^{-2x}$ is a solution of the complementary equation $y' + 2y = 0$. Therefore we seek solutions of (2.1.18) in the form $y = ue^{-2x}$, so that

$$y' = u'e^{-2x} - 2ue^{-2x} \quad \text{and} \quad y' + 2y = u'e^{-2x} - 2ue^{-2x} + 2ue^{-2x} = u'e^{-2x}. \quad (2.1.19)$$

Therefore y is a solution of (2.1.18) if and only if

$$u'e^{-2x} = x^3 e^{-2x} \quad \text{or, equivalently,} \quad u' = x^3.$$

Therefore

$$u = \frac{x^4}{4} + c,$$

and

$$y = ue^{-2x} = e^{-2x} \left(\frac{x^4}{4} + c \right)$$

is the general solution of (2.1.18).

Figure 2.1.3 shows a direction field and some integral curves for (2.1.18).

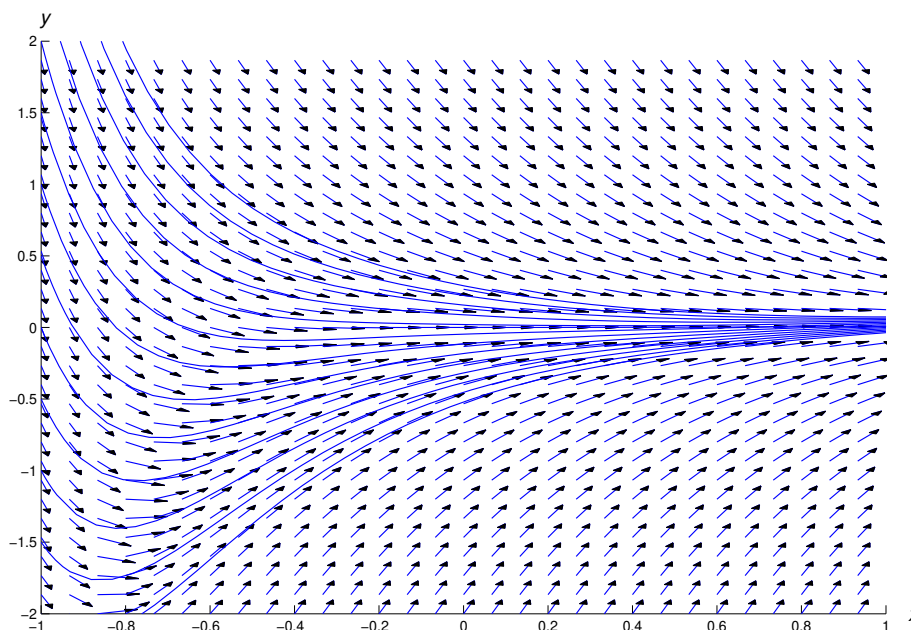


Figure 2.1.3 A direction field and integral curves for $y' + 2y = x^3 e^{-2x}$

Example 2.1.6

(a) Find the general solution of

$$y' + (\cot x)y = x \csc x. \quad (2.1.20)$$

(b) Solve the initial value problem

$$y' + (\cot x)y = x \csc x, \quad y(\pi/2) = 1. \quad (2.1.21)$$

Solution (a) Here $p(x) = \cot x$ and $f(x) = x \csc x$ are both continuous except at the points $x = r\pi$, where r is an integer. Therefore we seek solutions of (2.1.20) on the intervals $(r\pi, (r+1)\pi)$. We need a nontrivial solution y_1 of the complementary equation; thus, y_1 must satisfy $y_1' + (\cot x)y_1 = 0$, which we rewrite as

$$\frac{y_1'}{y_1} = -\cot x = -\frac{\cos x}{\sin x}. \quad (2.1.22)$$

Integrating this yields

$$\ln |y_1| = -\ln |\sin x|,$$

where we take the constant of integration to be zero since we need only *one* function that satisfies (2.1.22). Clearly $y_1 = 1/\sin x$ is a suitable choice. Therefore we seek solutions of (2.1.20) in the form

$$y = \frac{u}{\sin x},$$

so that

$$y' = \frac{u'}{\sin x} - \frac{u \cos x}{\sin^2 x} \quad (2.1.23)$$

and

$$\begin{aligned}
 y' + (\cot x)y &= \frac{u'}{\sin x} - \frac{u \cos x}{\sin^2 x} + \frac{u \cot x}{\sin x} \\
 &= \frac{u'}{\sin x} - \frac{u \cos x}{\sin^2 x} + \frac{u \cos x}{\sin^2 x} \\
 &= \frac{u'}{\sin x}.
 \end{aligned} \tag{2.1.24}$$

Therefore y is a solution of (2.1.20) if and only if

$$u'/\sin x = x \csc x = x/\sin x \quad \text{or, equivalently,} \quad u' = x.$$

Integrating this yields

$$u = \frac{x^2}{2} + c, \quad \text{and} \quad y = \frac{u}{\sin x} = \frac{x^2}{2 \sin x} + \frac{c}{\sin x} \tag{2.1.25}$$

is the general solution of (2.1.20) on every interval $(r\pi, (r+1)\pi)$ (r =integer).

Solution (b) Imposing the initial condition $y(\pi/2) = 1$ in (2.1.25) yields

$$1 = \frac{\pi^2}{8} + c \quad \text{or} \quad c = 1 - \frac{\pi^2}{8}.$$

Thus,

$$y = \frac{x^2}{2 \sin x} + \frac{(1 - \pi^2/8)}{\sin x}$$

is a solution of (2.1.21). The interval of validity of this solution is $(0, \pi)$; Figure 2.1.4 shows its graph.

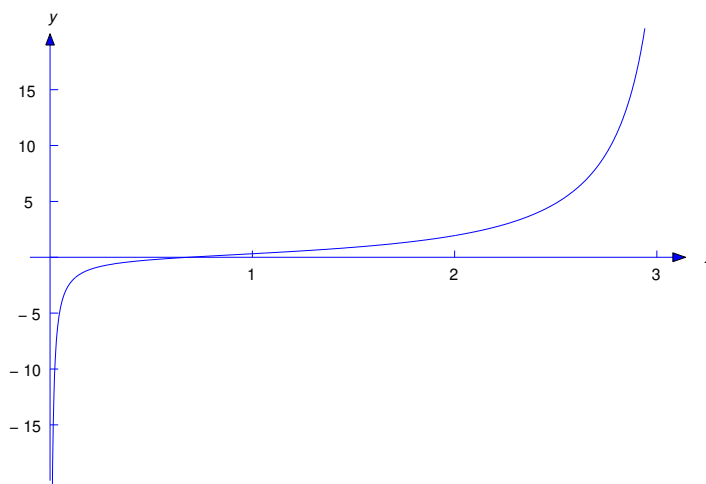


Figure 2.1.4 Solution of $y' + (\cot x)y = x \csc x$, $y(\pi/2) = 1$

Remark: It wasn't necessary to do the computations (2.1.23) and (2.1.24) in Example 2.1.6, since we showed in the discussion preceding Example 2.1.5 that if $y = uy_1$ where $y_1' + p(x)y_1 = 0$, then $y' + p(x)y = u'y_1$. We did these computations so you would see this happen in this specific example. We recommend that you include these “unnecessary” computations in doing exercises, until you're confident that you really understand the method. After that, omit them.

Example 2.1.7

(a) Find the general solution of

$$y' - 2xy = 1.$$

(b) Solve the initial value problem

$$y' - 2xy = 1, \quad y(0) = y_0. \quad (2.1.28)$$

Solution (a) To apply variation of parameters, we need a nontrivial solution y_1 of the complementary equation; thus, $y_1' - 2xy_1 = 0$, which we rewrite as

$$\frac{y_1'}{y_1} = 2x.$$

Integrating this and taking the constant of integration to be zero yields

$$\ln |y_1| = x^2, \quad \text{so} \quad |y_1| = e^{x^2}.$$

We choose $y_1 = e^{x^2}$ and seek solutions of (2.1.28) in the form $y = ue^{x^2}$, where

$$u'e^{x^2} = 1, \quad \text{so} \quad u' = e^{-x^2}.$$

Therefore

$$u = c + \int e^{-x^2} dx,$$

but we can't simplify the integral on the right because there's no elementary function with derivative equal to e^{-x^2} . Therefore the best available form for the general solution of (2.1.28) is

$$y = ue^{x^2} = e^{x^2} \left(c + \int e^{-x^2} dx \right). \quad (2.1.29)$$

Solution (b) Since the initial condition in (2.1.28) is imposed at $x_0 = 0$, it is convenient to rewrite (2.1.29) as

$$y = e^{x^2} \left(c + \int_0^x e^{-t^2} dt \right), \quad \text{since} \quad \int_0^0 e^{-t^2} dt = 0.$$

Setting $x = 0$ and $y = y_0$ here shows that $c = y_0$. Therefore the solution of the initial value problem is

$$y = e^{x^2} \left(y_0 + \int_0^x e^{-t^2} dt \right). \quad (2.1.30)$$

For a given value of y_0 and each fixed x , the integral on the right can be evaluated by numerical methods. An alternate procedure is to apply the numerical integration procedures discussed in Chapter 3 directly to the initial value problem (2.1.28). Figure 2.1.5 shows graphs of (2.1.30) for several values of y_0 .

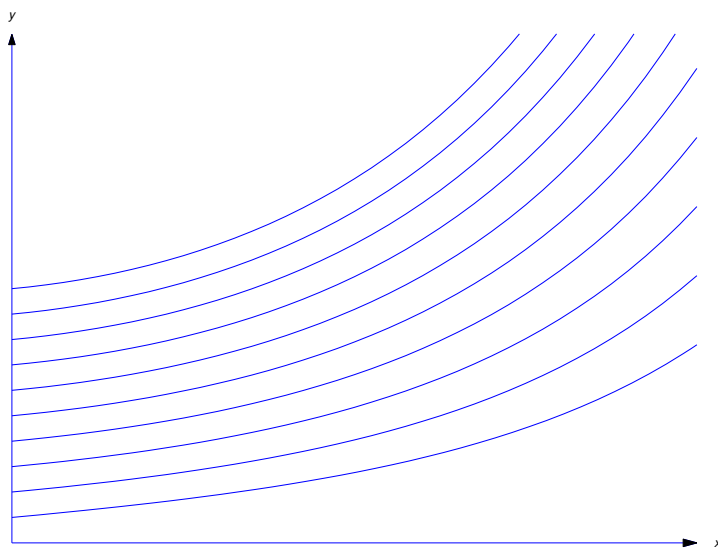


Figure 2.1.5 Solutions of $y' - 2xy = 1$, $y(0) = y_0$

Section 2.2

Example 2.2.1 Solve the equation

$$y' = x(1 + y^2).$$

Solution Separating variables yields

$$\frac{y'}{1 + y^2} = x.$$

Integrating yields

$$\tan^{-1} y = \frac{x^2}{2} + c$$

Therefore

$$y = \tan\left(\frac{x^2}{2} + c\right).$$

Example 2.2.2

(a) Solve the equation

$$y' = -\frac{x}{y}. \quad (2.2.4)$$

(b) Solve the initial value problem

$$y' = -\frac{x}{y}, \quad y(1) = 1. \quad (2.2.5)$$

(c) Solve the initial value problem

$$y' = -\frac{x}{y}, \quad y(1) = -2. \quad (2.2.6)$$

Solution (a) Separating variables in (2.2.4) yields

$$yy' = -x.$$

Integrating yields

$$\frac{y^2}{2} = -\frac{x^2}{2} + c, \quad \text{or, equivalently,} \quad x^2 + y^2 = 2c.$$

The last equation shows that c must be positive if y is to be a solution of (2.2.4) on an open interval. Therefore we let $2c = a^2$ (with $a > 0$) and rewrite the last equation as

$$x^2 + y^2 = a^2. \quad (2.2.7)$$

This equation has two differentiable solutions for y in terms of x :

$$y = \sqrt{a^2 - x^2}, \quad -a < x < a, \quad (2.2.8)$$

and

$$y = -\sqrt{a^2 - x^2}, \quad -a < x < a. \quad (2.2.9)$$

The solution curves defined by (2.2.8) are semicircles above the x -axis and those defined by (2.2.9) are semicircles below the x -axis (Figure 2.2.1).

Solution (b) The solution of (2.2.5) is positive when $x = 1$; hence, it is of the form (2.2.8). Substituting $x = 1$ and $y = 1$ into (2.2.7) to satisfy the initial condition yields $a^2 = 2$; hence, the solution of (2.2.5) is

$$y = \sqrt{2 - x^2}, \quad -\sqrt{2} < x < \sqrt{2}.$$

Solution (c) The solution of (2.2.6) is negative when $x = 1$ and is therefore of the form (2.2.9). Substituting $x = 1$ and $y = -2$ into (2.2.7) to satisfy the initial condition yields $a^2 = 5$. Hence, the solution of (2.2.6) is

$$y = -\sqrt{5 - x^2}, \quad -\sqrt{5} < x < \sqrt{5}.$$

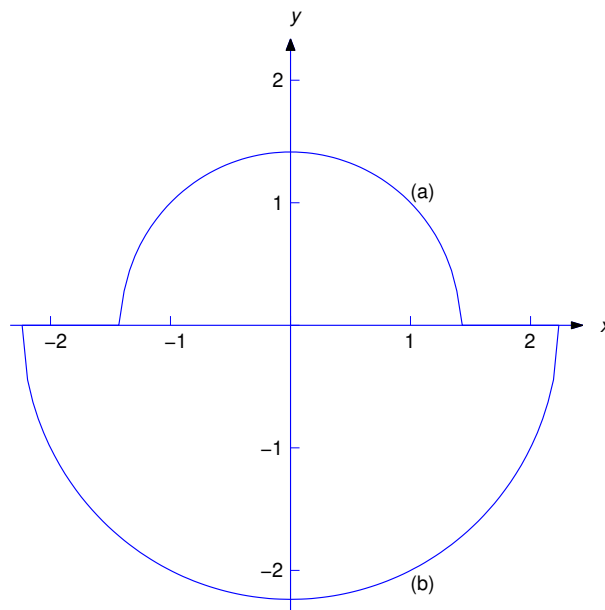


Figure 2.2.1 (a) $y = \sqrt{2 - x^2}$, $-\sqrt{2} < x < \sqrt{2}$; (b) $y = -\sqrt{5 - x^2}$, $-\sqrt{5} < x < \sqrt{5}$

Example 2.2.3**(a)** Find implicit solutions of

$$y' = \frac{2x + 1}{5y^4 + 1}. \quad (2.2.13)$$

(b) Find an implicit solution of

$$y' = \frac{2x + 1}{5y^4 + 1}, \quad y(2) = 1. \quad (2.2.14)$$

Solution (a) Separating variables yields

$$(5y^4 + 1)y' = 2x + 1.$$

Integrating yields the implicit solution

$$y^5 + y = x^2 + x + c \quad (2.2.15)$$

of (2.2.13).

Solution (b) Imposing the initial condition $y(2) = 1$ in (2.2.15) yields $1 + 1 = 4 + 2 + c$, so $c = -4$. Therefore

$$y^5 + y = x^2 + x - 4$$

is an implicit solution of the initial value problem (2.2.14). Although more than one differentiable function $y = y(x)$ satisfies (2.2.15) near $x = 2$, it can be shown that there's only one such function that satisfies the initial condition $y(2) = 1$.

Figure 2.2.2 shows a direction field and some integral curves for (2.2.13).

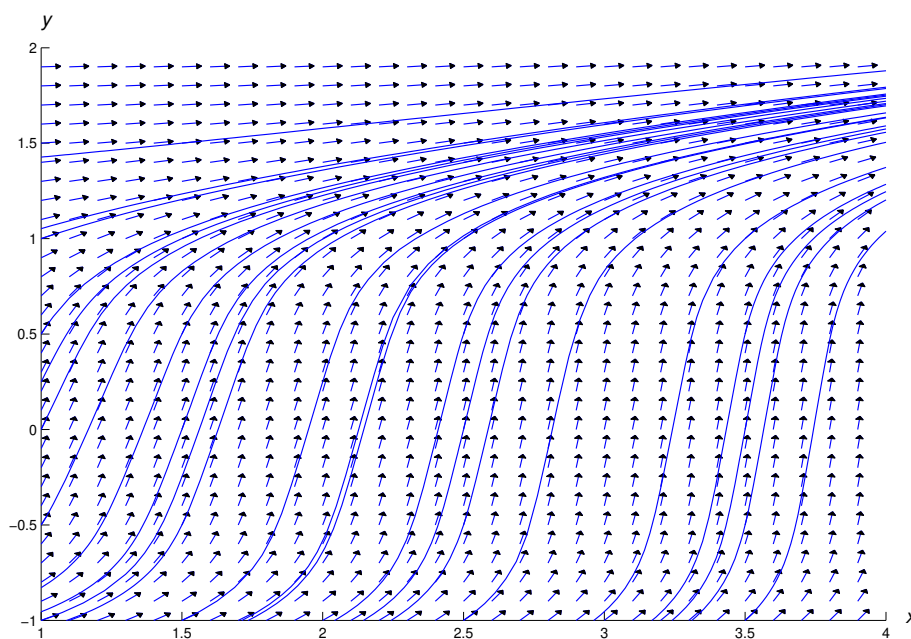


Figure 2.2.2 A direction field and integral curves for $y' = \frac{2x + 1}{5y^4 + 1}$

Example 2.2.4 Find all solutions of

$$y' = 2xy^2. \quad (2.2.16)$$

Solution Here we must divide by $p(y) = y^2$ to separate variables. This isn't legitimate if y is a solution of (2.2.16) that equals zero for some value of x . One such solution can be found by inspection: $y \equiv 0$. Now suppose y is a solution of (2.2.16) that isn't identically zero. Since y is continuous there must be an interval on which y is never zero. Since division by y^2 is legitimate for x in this interval, we can separate variables in (2.2.16) to obtain

$$\frac{y'}{y^2} = 2x.$$

Integrating this yields

$$-\frac{1}{y} = x^2 + c,$$

which is equivalent to

$$y = -\frac{1}{x^2 + c}. \quad (2.2.17)$$

We've now shown that if y is a solution of (2.2.16) that is not identically zero, then y must be of the form (2.2.17). By substituting (2.2.17) into (2.2.16), you can verify that (2.2.17) is a solution of (2.2.16). Thus, solutions of (2.2.16) are $y \equiv 0$ and the functions of the form (2.2.17). Note that the solution $y \equiv 0$ isn't of the form (2.2.17) for any value of c .

Figure 2.2.3 shows a direction field and some integral curves for (2.2.16).

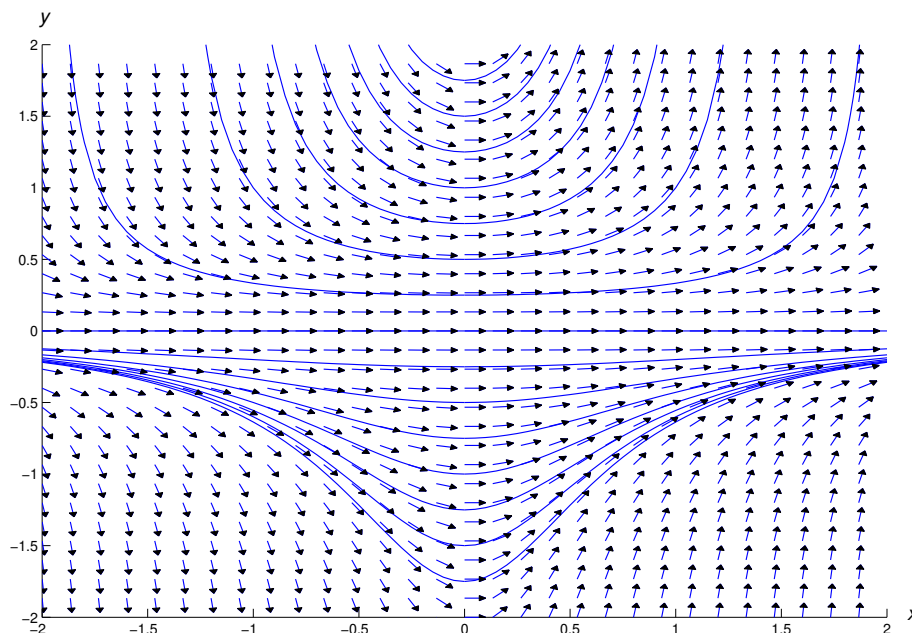


Figure 2.2.3 A direction field and integral curves for $y' = 2xy^2$

Example 2.2.5 Find all solutions of

$$y' = \frac{1}{2}x(1 - y^2). \quad (2.2.18)$$

Solution Here we must divide by $p(y) = 1 - y^2$ to separate variables. This isn't legitimate if y is a solution of (2.2.18) that equals ± 1 for some value of x . Two such solutions can be found by inspection: $y \equiv 1$ and $y \equiv -1$. Now suppose y is a solution of (2.2.18) such that $1 - y^2$ isn't identically zero. Since $1 - y^2$ is continuous there must be an interval on which $1 - y^2$ is never zero. Since division by $1 - y^2$ is legitimate for x in this interval, we can separate variables in (2.2.18) to obtain

$$\frac{2y'}{y^2 - 1} = -x.$$

A partial fraction expansion on the left yields

$$\left[\frac{1}{y-1} - \frac{1}{y+1} \right] y' = -x,$$

and integrating yields

$$\ln \left| \frac{y-1}{y+1} \right| = -\frac{x^2}{2} + k;$$

hence,

$$\left| \frac{y-1}{y+1} \right| = e^k e^{-x^2/2}.$$

Since $y(x) \neq \pm 1$ for x on the interval under discussion, the quantity $(y-1)/(y+1)$ can't change sign in this interval. Therefore we can rewrite the last equation as

$$\frac{y-1}{y+1} = c e^{-x^2/2},$$

where $c = \pm e^k$, depending upon the sign of $(y-1)/(y+1)$ on the interval. Solving for y yields

$$y = \frac{1 + c e^{-x^2/2}}{1 - c e^{-x^2/2}}. \quad (2.2.19)$$

We've now shown that if y is a solution of (2.2.18) that is not identically equal to ± 1 , then y must be as in (2.2.19). By substituting (2.2.19) into (2.2.18) you can verify that (2.2.19) is a solution of (2.2.18). Thus, the solutions of (2.2.18) are $y \equiv 1$, $y \equiv -1$ and the functions of the form (2.2.19). Note that the constant solution $y \equiv 1$ can be obtained from this formula by taking $c = 0$; however, the other constant solution, $y \equiv -1$, can't be obtained in this way.

Figure 2.2.4 shows a direction field and some integral curves for (2.2.18).

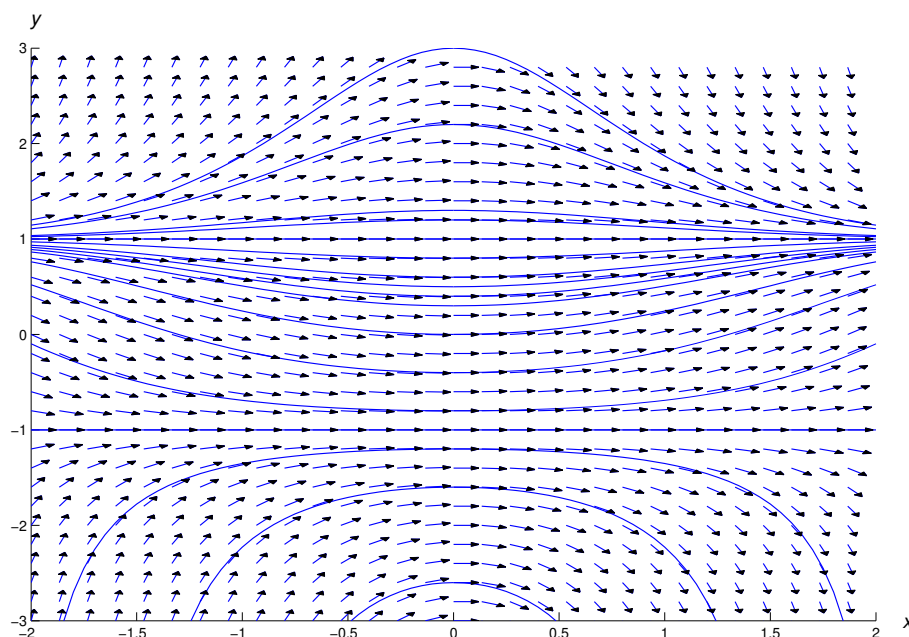


Figure 2.2.4 A direction field and integral curves for $y' = \frac{x(1 - y^2)}{2}$

Example 2.2.6 Solve the initial value problem

$$y' = 2xy^2, \quad y(0) = y_0$$

and determine the interval of validity of the solution.

Solution First suppose $y_0 \neq 0$. From Example 2.2.4, we know that y must be of the form

$$y = -\frac{1}{x^2 + c}. \quad (2.2.20)$$

Imposing the initial condition shows that $c = -1/y_0$. Substituting this into (2.2.20) and rearranging terms yields the solution

$$y = \frac{y_0}{1 - y_0 x^2}.$$

This is also the solution if $y_0 = 0$. If $y_0 < 0$, the denominator isn't zero for any value of x , so the solution is valid on $(-\infty, \infty)$. If $y_0 > 0$, the solution is valid only on $(-1/\sqrt{y_0}, 1/\sqrt{y_0})$.

Section 2.3

Example 2.3.1 Consider the initial value problem

$$y' = \frac{x^2 - y^2}{1 + x^2 + y^2}, \quad y(x_0) = y_0. \quad (2.3.3)$$

Since

$$f(x, y) = \frac{x^2 - y^2}{1 + x^2 + y^2} \quad \text{and} \quad f_y(x, y) = -\frac{2y(1 + 2x^2)}{(1 + x^2 + y^2)^2}$$

are continuous for all (x, y) , Theorem 2.3.1 implies that if (x_0, y_0) is arbitrary, then (2.3.3) has a unique solution on some open interval that contains x_0 .

Example 2.3.2 Consider the initial value problem

$$y' = \frac{x^2 - y^2}{x^2 + y^2}, \quad y(x_0) = y_0. \quad (2.3.4)$$

Here

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2} \quad \text{and} \quad f_y(x, y) = -\frac{4x^2y}{(x^2 + y^2)^2}$$

are continuous everywhere except at $(0, 0)$. If $(x_0, y_0) \neq (0, 0)$, there's an open rectangle R that contains (x_0, y_0) that does not contain $(0, 0)$. Since f and f_y are continuous on R , Theorem 2.3.1 implies that if $(x_0, y_0) \neq (0, 0)$ then (2.3.4) has a unique solution on some open interval that contains x_0 .

Example 2.3.3 Consider the initial value problem

$$y' = \frac{x + y}{x - y}, \quad y(x_0) = y_0. \quad (2.3.5)$$

Here

$$f(x, y) = \frac{x + y}{x - y} \quad \text{and} \quad f_y(x, y) = \frac{2x}{(x - y)^2}$$

are continuous everywhere except on the line $y = x$. If $y_0 \neq x_0$, there's an open rectangle R that contains (x_0, y_0) that does not intersect the line $y = x$. Since f and f_y are continuous on R , Theorem 2.3.1 implies that if $y_0 \neq x_0$, (2.3.5) has a unique solution on some open interval that contains x_0 .

Example 2.3.4 In Example 2.2.4 we saw that the solutions of

$$y' = 2xy^2 \quad (2.3.6)$$

are

$$y \equiv 0 \quad \text{and} \quad y = -\frac{1}{x^2 + c},$$

where c is an arbitrary constant. In particular, this implies that no solution of (2.3.6) other than $y \equiv 0$ can equal zero for any value of x . Show that Theorem 2.3.1(b) implies this.

Solution We'll obtain a contradiction by assuming that (2.3.6) has a solution y_1 that equals zero for some value of x , but isn't identically zero. If y_1 has this property, there's a point x_0 such that $y_1(x_0) = 0$, but $y_1(x) \neq 0$ for some value of x in every open interval that contains x_0 . This means that the initial value problem

$$y' = 2xy^2, \quad y(x_0) = 0 \quad (2.3.7)$$

has two solutions $y \equiv 0$ and $y = y_1$ that differ for some value of x on every open interval that contains x_0 . This contradicts Theorem 2.3.1(b), since in (2.3.6) the functions

$$f(x, y) = 2xy^2 \quad \text{and} \quad f_y(x, y) = 4xy$$

are both continuous for all (x, y) , which implies that (2.3.7) has a unique solution on some open interval that contains x_0 .

Example 2.3.5 Consider the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(x_0) = y_0. \quad (2.3.8)$$

- (a) For what points (x_0, y_0) does Theorem 2.3.1(a) imply that (2.3.8) has a solution?
- (b) For what points (x_0, y_0) does Theorem 2.3.1(b) imply that (2.3.8) has a unique solution on some open interval that contains x_0 ?

Solution (a) Since

$$f(x, y) = \frac{10}{3}xy^{2/5}$$

is continuous for all (x, y) , Theorem 2.3.1 implies that (2.3.8) has a solution for every (x_0, y_0) .

Solution (b) Here

$$f_y(x, y) = \frac{4}{3}xy^{-3/5}$$

is continuous for all (x, y) with $y \neq 0$. Therefore, if $y_0 \neq 0$ there's an open rectangle on which both f and f_y are continuous, and Theorem 2.3.1 implies that (2.3.8) has a unique solution on some open interval that contains x_0 .

If $y = 0$ then $f_y(x, y)$ is undefined, and therefore discontinuous; hence, Theorem 2.3.1 does not apply to (2.3.8) if $y_0 = 0$.

Example 2.3.6 Example 2.3.5 leaves open the possibility that the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(0) = 0 \quad (2.3.9)$$

has more than one solution on every open interval that contains $x_0 = 0$. Show that this is true.

Solution By inspection, $y \equiv 0$ is a solution of the differential equation

$$y' = \frac{10}{3}xy^{2/5}. \quad (2.3.10)$$

Since $y \equiv 0$ satisfies the initial condition $y(0) = 0$, it's a solution of (2.3.9).

Now suppose y is a solution of (2.3.10) that isn't identically zero. Separating variables in (2.3.10) yields

$$y^{-2/5}y' = \frac{10}{3}x$$

on any open interval where y has no zeros. Integrating this and rewriting the arbitrary constant as $5c/3$ yields

$$\frac{5}{3}y^{3/5} = \frac{5}{3}(x^2 + c).$$

Therefore

$$y = (x^2 + c)^{5/3}. \quad (2.3.11)$$

Since we divided by y to separate variables in (2.3.10), our derivation of (2.3.11) is legitimate only on open intervals where y has no zeros. However, (2.3.11) actually defines y for all x , and differentiating (2.3.11) shows that

$$y' = \frac{10}{3}x(x^2 + c)^{2/3} = \frac{10}{3}xy^{2/5}, \quad -\infty < x < \infty.$$

Therefore (2.3.11) satisfies (2.3.10) on $(-\infty, \infty)$ even if $c \leq 0$, so that $y(\sqrt{|c|}) = y(-\sqrt{|c|}) = 0$. In particular, taking $c = 0$ in (2.3.11) yields

$$y = x^{10/3}$$

as a second solution of (2.3.9). Both solutions are defined on $(-\infty, \infty)$, and they differ on every open interval that contains $x_0 = 0$ (see Figure 2.3.2.) In fact, there are *four* distinct solutions of (2.3.9) defined on $(-\infty, \infty)$ that differ from each other on every open interval that contains $x_0 = 0$. Can you identify the other two?

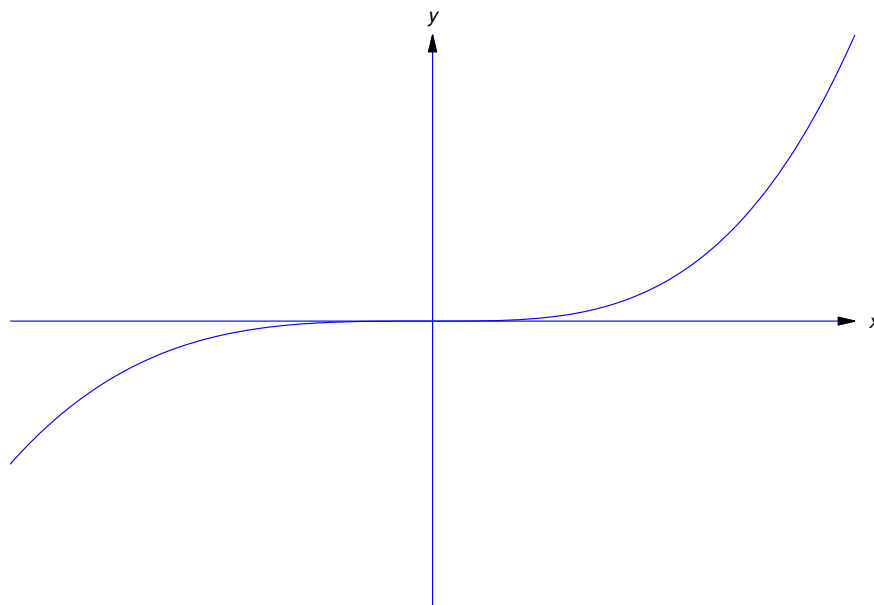


Figure 2.3.2 Two solutions ($y = 0$ and $y = x^{10/3}$) of (2.3.9) that differ on every interval containing $x_0 = 0$

Example 2.3.7 From Example 2.3.5, the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(0) = -1 \tag{2.3.12}$$

has a unique solution on some open interval that contains $x_0 = 0$. Find a solution and determine the largest open interval (a, b) on which it's unique.

Solution Let y be any solution of (2.3.12). Because of the initial condition $y(0) = -1$ and the continuity of y , there's an open interval I that contains $x_0 = 0$ on which y has no zeros, and is consequently of the form (2.3.11). Setting $x = 0$ and $y = -1$ in (2.3.11) yields $c = -1$, so

$$y = (x^2 - 1)^{5/3} \tag{2.3.13}$$

for x in I . Therefore every solution of (2.3.12) differs from zero and is given by (2.3.13) on $(-1, 1)$; that is, (2.3.13) is the unique solution of (2.3.12) on $(-1, 1)$. This is the largest open interval on which (2.3.12) has a unique solution. To see this, note that (2.3.13) is a solution of (2.3.12) on $(-\infty, \infty)$. From Exercise 2.3.2.3.15, there are infinitely many other solutions of

(2.3.12) that differ from (2.3.13) on every open interval larger than $(-1, 1)$. One such solution is

$$y = \begin{cases} (x^2 - 1)^{5/3}, & -1 \leq x \leq 1, \\ 0, & |x| > 1. \end{cases}$$

(Figure 2.3.3).

Example 2.3.8 From Example 2.3.5, the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(0) = 1 \quad (2.3.14)$$

has a unique solution on some open interval that contains $x_0 = 0$. Find the solution and determine the largest open interval on which it's unique.

Solution Let y be any solution of (2.3.14). Because of the initial condition $y(0) = 1$ and the continuity of y , there's an open interval I that contains $x_0 = 0$ on which y has no zeros, and is consequently of the form (2.3.11). Setting $x = 0$ and $y = 1$ in (2.3.11) yields $c = 1$, so

$$y = (x^2 + 1)^{5/3} \quad (2.3.15)$$

for x in I . Therefore every solution of (2.3.14) differs from zero and is given by (2.3.15) on $(-\infty, \infty)$; that is, (2.3.15) is the unique solution of (2.3.14) on $(-\infty, \infty)$. Figure 2.3.4 shows the graph of this solution.

Section 2.4

Example 2.4.1 Solve the Bernoulli equation

$$y' - y = xy^2. \quad (2.4.3)$$

Solution Since $y_1 = e^x$ is a solution of $y' - y = 0$, we look for solutions of (2.4.3) in the form $y = ue^x$, where

$$u'e^x = xu^2e^{2x} \quad \text{or, equivalently,} \quad u' = xu^2e^x.$$

Separating variables yields

$$\frac{u'}{u^2} = xe^x,$$

and integrating yields

$$-\frac{1}{u} = (x - 1)e^x + c.$$

Hence,

$$u = -\frac{1}{(x - 1)e^x + c}$$

and

$$y = -\frac{1}{x - 1 + ce^{-x}}.$$

Figure 2.4.1 shows a direction field and some integral curves of (2.4.3).

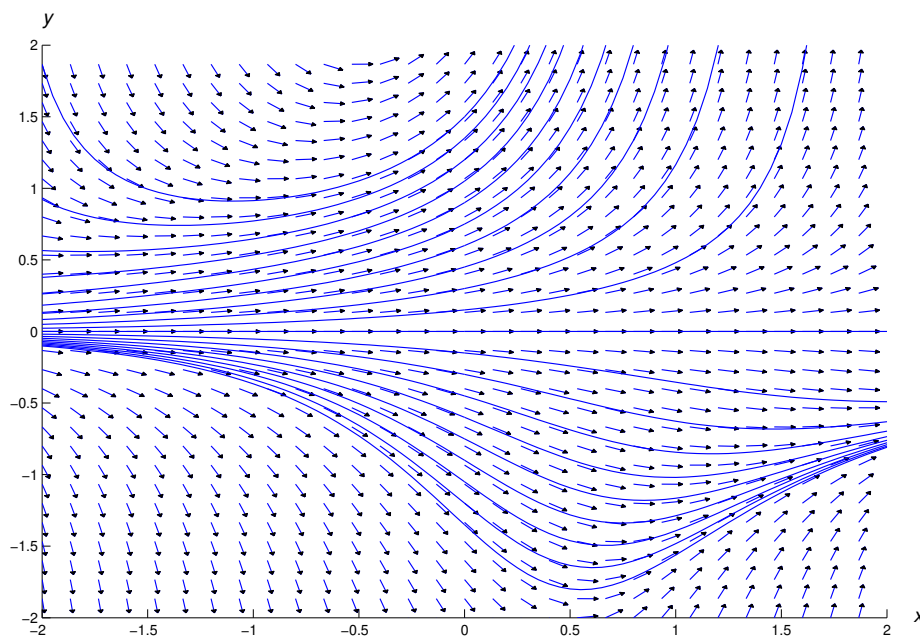


Figure 2.4.1 A direction field and integral curves for $y' - y = xy^2$

Example 2.4.2 Solve

$$y' = \frac{y + xe^{-y/x}}{x}. \quad (2.4.8)$$

Solution Substituting $y = ux$ into (2.4.8) yields

$$u'x + u = \frac{ux + xe^{-ux/x}}{x} = u + e^{-u}.$$

Simplifying and separating variables yields

$$e^u u' = \frac{1}{x}.$$

Integrating yields $e^u = \ln|x| + c$. Therefore $u = \ln(\ln|x| + c)$ and $y = ux = x \ln(\ln|x| + c)$.

Figure 2.4.2 shows a direction field and integral curves for (2.4.8).

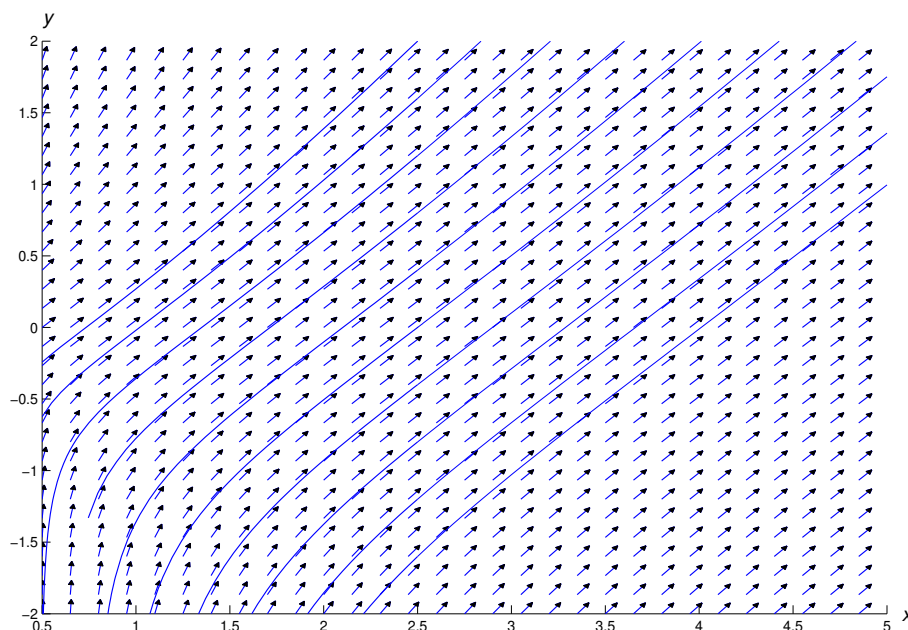


Figure 2.4.2 A direction field and some integral curves for $y' = \frac{y + xe^{-y/x}}{x}$

Example 2.4.3

(a) Solve

$$x^2 y' = y^2 + xy - x^2. \quad (2.4.9)$$

(b) Solve the initial value problem

$$x^2 y' = y^2 + xy - x^2, \quad y(1) = 2. \quad (2.4.10)$$

Solution (a) We first find solutions of (2.4.9) on open intervals that don't contain $x = 0$. We can rewrite (2.4.9) as

$$y' = \frac{y^2 + xy - x^2}{x^2}$$

for x in any such interval. Substituting $y = ux$ yields

$$u'x + u = \frac{(ux)^2 + x(ux) - x^2}{x^2} = u^2 + u - 1,$$

so

$$u'x = u^2 - 1. \quad (2.4.11)$$

By inspection this equation has the constant solutions $u \equiv 1$ and $u \equiv -1$. Therefore $y = x$ and $y = -x$ are solutions of (2.4.9). If u is a solution of (2.4.11) that doesn't assume the values ± 1 on some interval, separating variables yields

$$\frac{u'}{u^2 - 1} = \frac{1}{x},$$

or, after a partial fraction expansion,

$$\frac{1}{2} \left[\frac{1}{u-1} - \frac{1}{u+1} \right] u' = \frac{1}{x}.$$

Multiplying by 2 and integrating yields

$$\ln \left| \frac{u-1}{u+1} \right| = 2 \ln |x| + k,$$

or

$$\left| \frac{u-1}{u+1} \right| = e^k x^2,$$

which holds if

$$\frac{u-1}{u+1} = cx^2 \quad (2.4.12)$$

where c is an arbitrary constant. Solving for u yields

$$u = \frac{1 + cx^2}{1 - cx^2}.$$

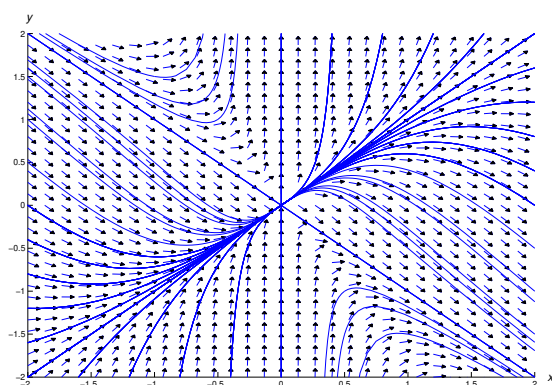


Figure 2.4.3 A direction field and integral curves for $x^2 y' = y^2 + xy - x^2$

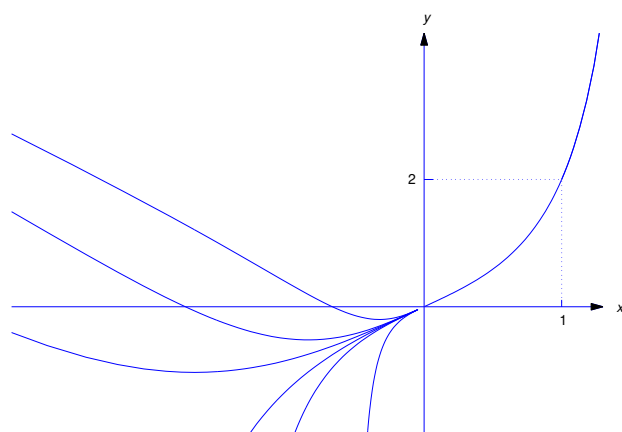


Figure 2.4.4 Solutions of $x^2 y' = y^2 + xy - x^2$, $y(1) = 2$

Therefore

$$y = ux = \frac{x(1 + cx^2)}{1 - cx^2} \quad (2.4.13)$$

is a solution of (2.4.9) for any choice of the constant c . Setting $c = 0$ in (2.4.13) yields the solution $y = x$. However, the solution $y = -x$ can't be obtained from (2.4.13). Thus, the solutions of (2.4.9) on intervals that don't contain $x = 0$ are $y = -x$ and functions of the form (2.4.13).

The situation is more complicated if $x = 0$ is in the open interval. First, note that $y = -x$ satisfies (2.4.9) on $(-\infty, \infty)$. If c_1 and c_2 are arbitrary constants, the function

$$y = \begin{cases} \frac{x(1 + c_1 x^2)}{1 - c_1 x^2}, & a < x < 0, \\ \frac{x(1 + c_2 x^2)}{1 - c_2 x^2}, & 0 \leq x < b, \end{cases} \quad (2.4.14)$$

is a solution of (2.4.9) on (a, b) , where

$$a = \begin{cases} -\frac{1}{\sqrt{c_1}} & \text{if } c_1 > 0, \\ -\infty & \text{if } c_1 \leq 0, \end{cases} \quad \text{and} \quad b = \begin{cases} \frac{1}{\sqrt{c_2}} & \text{if } c_2 > 0, \\ \infty & \text{if } c_2 \leq 0. \end{cases}$$

We leave it to you to verify this. To do so, note that if y is any function of the form (2.4.13) then $y(0) = 0$ and $y'(0) = 1$.

Figure 2.4.3 shows a direction field and some integral curves for (2.4.9).

Solution (b) We could obtain c by imposing the initial condition $y(1) = 2$ in (2.4.13), and then solving for c . However, it's easier to use (2.4.12). Since $u = y/x$, the initial condition $y(1) = 2$ implies that $u(1) = 2$. Substituting this into (2.4.12) yields $c = 1/3$. Hence, the solution of (2.4.10) is

$$y = \frac{x(1 + x^2/3)}{1 - x^2/3}.$$

The interval of validity of this solution is $(-\sqrt{3}, \sqrt{3})$. However, the largest interval on which (2.4.10) has a unique solution is $(0, \sqrt{3})$. To see this, note from (2.4.14) that any function of the form

$$y = \begin{cases} \frac{x(1 + cx^2)}{1 - cx^2}, & a < x \leq 0, \\ \frac{x(1 + x^2/3)}{1 - x^2/3}, & 0 \leq x < \sqrt{3}, \end{cases} \quad (2.4.15)$$

is a solution of (2.4.10) on $(a, \sqrt{3})$, where $a = -1/\sqrt{c}$ if $c > 0$ or $a = -\infty$ if $c \leq 0$. (Why doesn't this contradict Theorem 2.3.1?)

Figure 2.4.4 shows several solutions of the initial value problem (2.4.10). Note that these solutions coincide on $(0, \sqrt{3})$.

Section 2.5

Example 2.5.1 Show that

$$x^4y^3 + x^2y^5 + 2xy = c \quad (2.5.4)$$

is an implicit solution of

$$(4x^3y^3 + 2xy^5 + 2y) dx + (3x^4y^2 + 5x^2y^4 + 2x) dy = 0. \quad (2.5.5)$$

Solution Regarding y as a function of x and differentiating (2.5.4) implicitly with respect to x yields

$$(4x^3y^3 + 2xy^5 + 2y) + (3x^4y^2 + 5x^2y^4 + 2x) \frac{dy}{dx} = 0.$$

Similarly, regarding x as a function of y and differentiating (2.5.4) implicitly with respect to y yields

$$(4x^3y^3 + 2xy^5 + 2y) \frac{dx}{dy} + (3x^4y^2 + 5x^2y^4 + 2x) = 0.$$

Therefore (2.5.4) is an implicit solution of (2.5.5) in either of its two possible interpretations.

■

Example 2.5.2 Show that the equation

$$3x^2y \, dx + 4x^3 \, dy = 0$$

is not exact on any open rectangle.

Solution Here

$$M(x, y) = 3x^2y \quad \text{and} \quad N(x, y) = 4x^3$$

so

$$M_y(x, y) = 3x^2 \quad \text{and} \quad N_x(x, y) = 12x^2.$$

Therefore $M_y = N_x$ on the line $x = 0$, but not on any open rectangle, so there's no function F such that $F_x(x, y) = M(x, y)$ and $F_y(x, y) = N(x, y)$ for all (x, y) on any open rectangle. ■

Example 2.5.3 Solve

$$(4x^3y^3 + 3x^2) \, dx + (3x^4y^2 + 6y^2) \, dy = 0. \quad (2.5.13)$$

Solution (Method 1) Here

$$M(x, y) = 4x^3y^3 + 3x^2, \quad N(x, y) = 3x^4y^2 + 6y^2,$$

and

$$M_y(x, y) = N_x(x, y) = 12x^3y^2$$

for all (x, y) . Therefore Theorem 2.5.2 implies that there's a function F such that

$$F_x(x, y) = M(x, y) = 4x^3y^3 + 3x^2 \quad (2.5.14)$$

and

$$F_y(x, y) = N(x, y) = 3x^4y^2 + 6y^2 \quad (2.5.15)$$

for all (x, y) . To find F , we integrate (2.5.14) with respect to x to obtain

$$F(x, y) = x^4y^3 + x^3 + \phi(y), \quad (2.5.16)$$

where $\phi(y)$ is the “constant” of integration. (Here ϕ is “constant” in that it's independent of x , the variable of integration.) If ϕ is any differentiable function of y then F satisfies (2.5.14). To determine ϕ so that F also satisfies (2.5.15), assume that ϕ is differentiable and differentiate F with respect to y . This yields

$$F_y(x, y) = 3x^4y^2 + \phi'(y).$$

Comparing this with (2.5.15) shows that

$$\phi'(y) = 6y^2.$$

We integrate this with respect to y and take the constant of integration to be zero because we're interested only in finding *some* F that satisfies (2.5.14) and (2.5.15). This yields

$$\phi(y) = 2y^3.$$

Substituting this into (2.5.16) yields

$$F(x, y) = x^4y^3 + x^3 + 2y^3. \quad (2.5.17)$$

Now Theorem 2.5.1 implies that

$$x^4 y^3 + x^3 + 2y^3 = c$$

is an implicit solution of (2.5.13). Solving this for y yields the explicit solution

$$y = \left(\frac{c - x^3}{2 + x^4} \right)^{1/3}.$$

Solution (Method 2) Instead of first integrating (2.5.14) with respect to x , we could begin by integrating (2.5.15) with respect to y to obtain

$$F(x, y) = x^4 y^3 + 2y^3 + \psi(x), \quad (2.5.18)$$

where ψ is an arbitrary function of x . To determine ψ , we assume that ψ is differentiable and differentiate F with respect to x , which yields

$$F_x(x, y) = 4x^3 y^3 + \psi'(x).$$

Comparing this with (2.5.14) shows that

$$\psi'(x) = 3x^2.$$

Integrating this and again taking the constant of integration to be zero yields

$$\psi(x) = x^3.$$

Substituting this into (2.5.18) yields (2.5.17).

Figure 2.5.1 shows a direction field and some integral curves of (2.5.13).

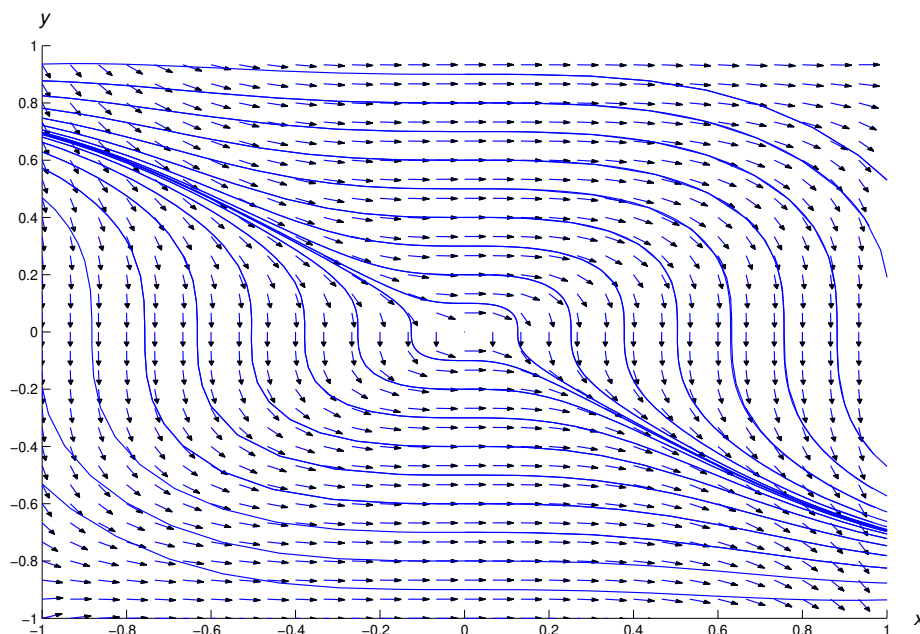


Figure 2.5.1 A direction field and integral curves for $(4x^3y^3 + 3x^2) dx + (3x^4y^2 + 6y^2) dy = 0$

Example 2.5.4 Solve the equation

$$(ye^{xy} \tan x + e^{xy} \sec^2 x) dx + xe^{xy} \tan x dy = 0. \quad (2.5.21)$$

Solution We leave it to you to check that $M_y = N_x$ on any open rectangle where $\tan x$ and $\sec x$ are defined. Here we must find a function F such that

$$F_x(x, y) = ye^{xy} \tan x + e^{xy} \sec^2 x \quad (2.5.22)$$

and

$$F_y(x, y) = xe^{xy} \tan x. \quad (2.5.23)$$

It's difficult to integrate (2.5.22) with respect to x , but easy to integrate (2.5.23) with respect to y . This yields

$$F(x, y) = e^{xy} \tan x + \psi(x). \quad (2.5.24)$$

Differentiating this with respect to x yields

$$F_x(x, y) = ye^{xy} \tan x + e^{xy} \sec^2 x + \psi'(x).$$

Comparing this with (2.5.22) shows that $\psi'(x) = 0$. Hence, ψ is a constant, which we can take to be zero in (2.5.24), and

$$e^{xy} \tan x = c$$

is an implicit solution of (2.5.21). ■

Example 2.5.5 Verify that the equation

$$3x^2y^2 dx + 6x^3y dy = 0 \quad (2.5.25)$$

is not exact, and show that the procedure for solving exact equations fails when applied to (2.5.25).

Solution Here

$$M_y(x, y) = 6x^2y \quad \text{and} \quad N_x(x, y) = 18x^2y,$$

so (2.5.25) isn't exact. Nevertheless, let's try to find a function F such that

$$F_x(x, y) = 3x^2y^2 \quad (2.5.26)$$

and

$$F_y(x, y) = 6x^3y. \quad (2.5.27)$$

Integrating (2.5.26) with respect to x yields

$$F(x, y) = x^3y^2 + \phi(y),$$

and differentiating this with respect to y yields

$$F_y(x, y) = 2x^3y + \phi'(y).$$

For this equation to be consistent with (2.5.27),

$$6x^3y = 2x^3y + \phi'(y),$$

or

$$\phi'(y) = 4x^3y.$$

This is a contradiction, since ϕ' must be independent of x . Therefore the procedure fails.

Section 2.6

Example 2.6.1 Find an integrating factor for the equation

$$(2xy^3 - 2x^3y^3 - 4xy^2 + 2x) dx + (3x^2y^2 + 4y) dy = 0 \quad (2.6.13)$$

and solve the equation.

Solution In (2.6.13)

$$M = 2xy^3 - 2x^3y^3 - 4xy^2 + 2x, \quad N = 3x^2y^2 + 4y,$$

and

$$M_y - N_x = (6xy^2 - 6x^3y^2 - 8xy) - 6xy^2 = -6x^3y^2 - 8xy,$$

so (2.6.13) isn't exact. However,

$$\frac{M_y - N_x}{N} = -\frac{6x^3y^2 + 8xy}{3x^2y^2 + 4y} = -2x$$

is independent of y , so Theorem 2.6.1(a) applies with $p(x) = -2x$. Since

$$\int p(x) dx = -\int 2x dx = -x^2,$$

$\mu(x) = e^{-x^2}$ is an integrating factor. Multiplying (2.6.13) by μ yields the exact equation

$$e^{-x^2}(2xy^3 - 2x^3y^3 - 4xy^2 + 2x) dx + e^{-x^2}(3x^2y^2 + 4y) dy = 0. \quad (2.6.14)$$

To solve this equation, we must find a function F such that

$$F_x(x, y) = e^{-x^2}(2xy^3 - 2x^3y^3 - 4xy^2 + 2x) \quad (2.6.15)$$

and

$$F_y(x, y) = e^{-x^2}(3x^2y^2 + 4y). \quad (2.6.16)$$

Integrating (2.6.16) with respect to y yields

$$F(x, y) = e^{-x^2}(x^2y^3 + 2y^2) + \psi(x). \quad (2.6.17)$$

Differentiating this with respect to x yields

$$F_x(x, y) = e^{-x^2}(2xy^3 - 2x^3y^3 - 4xy^2) + \psi'(x).$$

Comparing this with (2.6.15) shows that $\psi'(x) = 2xe^{-x^2}$; therefore, we can let $\psi(x) = -e^{-x^2}$ in (2.6.17) and conclude that

$$e^{-x^2}(y^2(x^2y + 2) - 1) = c$$

is an implicit solution of (2.6.14). It is also an implicit solution of (2.6.13).

Figure 2.6.1 shows a direction field and some integral curves for (2.6.13).

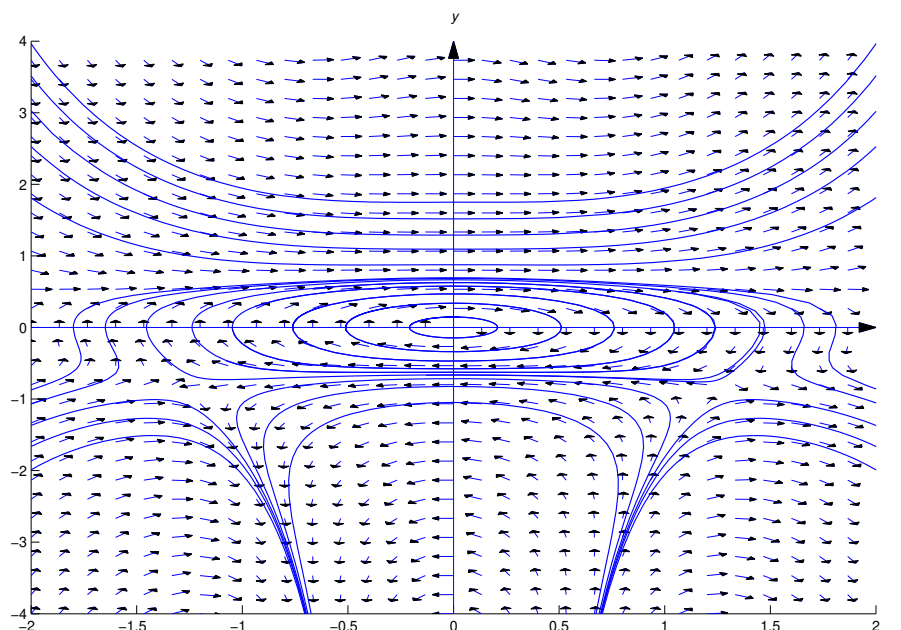


Figure 2.6.1 A direction field and integral curves for $(2xy^3 - 2x^3y^3 - 4xy^2 + 2x) dx + (3x^2y^2 + 4y) dy = 0$

Example 2.6.2 Find an integrating factor for

$$2xy^3 dx + (3x^2y^2 + x^2y^3 + 1) dy = 0 \quad (2.6.18)$$

and solve the equation.

Solution In (2.6.18),

$$M = 2xy^3, \quad N = 3x^2y^2 + x^2y^3 + 1,$$

and

$$M_y - N_x = 6xy^2 - (6xy^2 + 2xy^3) = -2xy^3,$$

so (2.6.18) isn't exact. Moreover,

$$\frac{M_y - N_x}{N} = -\frac{2xy^3}{3x^2y^2 + x^2y^3 + 1}$$

is not independent of y , so Theorem 2.6.1(a) does not apply. However, Theorem 2.6.1(b) does apply, since

$$\frac{N_x - M_y}{M} = \frac{2xy^3}{2xy^3} = 1$$

is independent of x , so we can take $q(y) = 1$. Since

$$\int q(y) dy = \int dy = y,$$

$\mu(y) = e^y$ is an integrating factor. Multiplying (2.6.18) by μ yields the exact equation

$$2xy^3e^y dx + (3x^2y^2 + x^2y^3 + 1)e^y dy = 0. \quad (2.6.19)$$

To solve this equation, we must find a function F such that

$$F_x(x, y) = 2xy^3e^y \quad (2.6.20)$$

and

$$F_y(x, y) = (3x^2y^2 + x^2y^3 + 1)e^y. \quad (2.6.21)$$

Integrating (2.6.20) with respect to x yields

$$F(x, y) = x^2y^3e^y + \phi(y). \quad (2.6.22)$$

Differentiating this with respect to y yields

$$F_y = (3x^2y^2 + x^2y^3)e^y + \phi'(y),$$

and comparing this with (2.6.21) shows that $\phi'(y) = e^y$. Therefore we set $\phi(y) = e^y$ in (2.6.22) and conclude that

$$(x^2y^3 + 1)e^y = c$$

is an implicit solution of (2.6.19). It is also an implicit solution of (2.6.18). Figure 2.6.2 shows a direction field and some integral curves for (2.6.18). ■

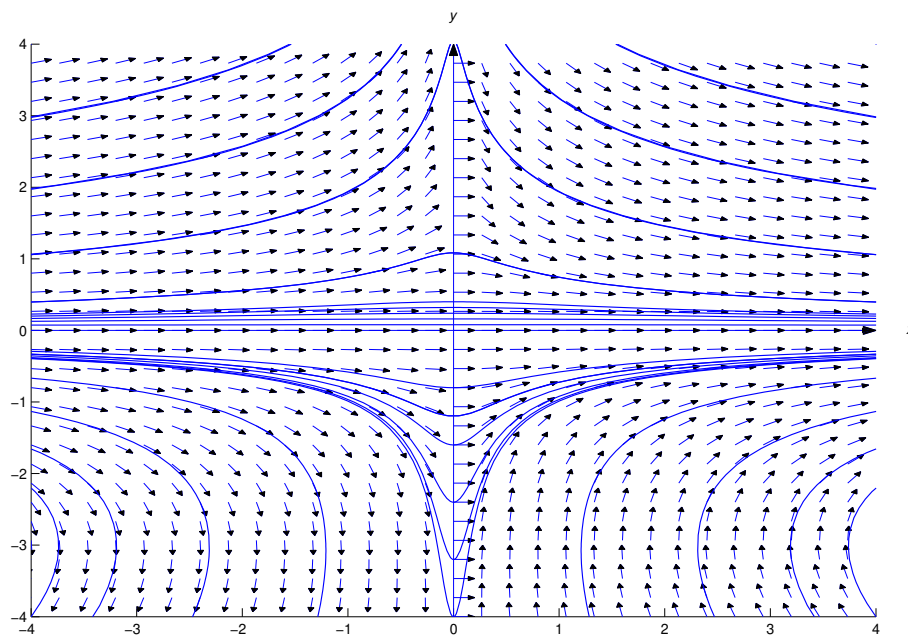


Figure 2.6.2 A direction field and integral curves for $2xy^3e^y dx + (3x^2y^2 + x^2y^3 + 1)e^y dy = 0$

Example 2.6.3 Find an integrating factor for

$$(3xy + 6y^2) dx + (2x^2 + 9xy) dy = 0 \quad (2.6.23)$$

and solve the equation.

Solution In (2.6.23)

$$M = 3xy + 6y^2, \quad N = 2x^2 + 9xy,$$

and

$$M_y - N_x = (3x + 12y) - (4x + 9y) = -x + 3y.$$

Therefore

$$\frac{M_y - N_x}{M} = \frac{-x + 3y}{3xy + 6y^2} \quad \text{and} \quad \frac{N_x - M_y}{N} = \frac{x - 3y}{2x^2 + 9xy},$$

so Theorem 2.6.1 does not apply. Following the more general argument that led to Theorem 2.6.1, we look for functions $p = p(x)$ and $q = q(y)$ such that

$$M_y - N_x = p(x)N - q(y)M;$$

that is,

$$-x + 3y = p(x)(2x^2 + 9xy) - q(y)(3xy + 6y^2).$$

Since the left side contains only first degree terms in x and y , we rewrite this equation as

$$xp(x)(2x + 9y) - yq(y)(3x + 6y) = -x + 3y.$$

This will be an identity if

$$xp(x) = A \quad \text{and} \quad yq(y) = B, \tag{2.6.24}$$

where A and B are constants such that

$$-x + 3y = A(2x + 9y) - B(3x + 6y),$$

or, equivalently,

$$-x + 3y = (2A - 3B)x + (9A - 6B)y.$$

Equating the coefficients of x and y on both sides shows that the last equation holds for all (x, y) if

$$\begin{aligned} 2A - 3B &= -1 \\ 9A - 6B &= 3, \end{aligned}$$

which has the solution $A = 1, B = 1$. Therefore (2.6.24) implies that

$$p(x) = \frac{1}{x} \quad \text{and} \quad q(y) = \frac{1}{y}.$$

Since

$$\int p(x) dx = \ln |x| \quad \text{and} \quad \int q(y) dy = \ln |y|,$$

we can let $P(x) = x$ and $Q(y) = y$; hence, $\mu(x, y) = xy$ is an integrating factor. Multiplying (2.6.23) by μ yields the exact equation

$$(3x^2y^2 + 6xy^3) dx + (2x^3y + 9x^2y^2) dy = 0.$$

We leave it to you to use the method of Section 2.5 to show that this equation has the implicit solution

$$x^3y^2 + 3x^2y^3 = c. \tag{2.6.25}$$

This is also an implicit solution of (2.6.23). Since $x \equiv 0$ and $y \equiv 0$ satisfy (2.6.25), you should check to see that $x \equiv 0$ and $y \equiv 0$ are also solutions of (2.6.23). (Why is it necessary to check this?)

Figure 2.6.3 shows a direction field and integral curves for (2.6.23).

Example 2.6.4 The separable equation

$$-y \, dx + (x + x^6) \, dy = 0 \quad (2.6.26)$$

can be converted to the exact equation

$$-\frac{dx}{x + x^6} + \frac{dy}{y} = 0 \quad (2.6.27)$$

by multiplying through by the integrating factor

$$\mu(x, y) = \frac{1}{y(x + x^6)}.$$

However, to solve (2.6.27) by the method of Section 2.5 we would have to evaluate the nasty integral

$$\int \frac{dx}{x + x^6}.$$

Instead, we solve (2.6.26) explicitly for y by finding an integrating factor of the form $\mu(x, y) = x^a y^b$.

Solution In (2.6.26)

$$M = -y, \quad N = x + x^6,$$

and

$$M_y - N_x = -1 - (1 + 6x^5) = -2 - 6x^5.$$

We look for functions $p = p(x)$ and $q = q(y)$ such that

$$M_y - N_x = p(x)N - q(y)M;$$

that is,

$$-2 - 6x^5 = p(x)(x + x^6) + q(y)y. \quad (2.6.28)$$

The right side will contain the term $-6x^5$ if $p(x) = -6/x$. Then (2.6.28) becomes

$$-2 - 6x^5 = -6 - 6x^5 + q(y)y,$$

so $q(y) = 4/y$. Since

$$\int p(x) \, dx = - \int \frac{6}{x} \, dx = -6 \ln |x| = \ln \frac{1}{x^6},$$

and

$$\int q(y) \, dy = \int \frac{4}{y} \, dy = 4 \ln |y| = \ln y^4,$$

we can take $P(x) = x^{-6}$ and $Q(y) = y^4$, which yields the integrating factor $\mu(x, y) = x^{-6}y^4$. Multiplying (2.6.26) by μ yields the exact equation

$$-\frac{y^5}{x^6} \, dx + \left(\frac{y^4}{x^5} + y^4 \right) \, dy = 0.$$

We leave it to you to use the method of the Section 2.5 to show that this equation has the implicit solution

$$\left(\frac{y}{x}\right)^5 + y^5 = k.$$

Solving for y yields

$$y = k^{1/5}x(1 + x^5)^{-1/5},$$

which we rewrite as

$$y = cx(1 + x^5)^{-1/5}$$

by renaming the arbitrary constant. This is also a solution of (2.6.26).

Figure 2.6.4 shows a direction field and some integral curves for (2.6.26).

Section 3.1

Example 3.1.1 Use Euler's method with $h = 0.1$ to find approximate values for the solution of the initial value problem

$$y' + 2y = x^3 e^{-2x}, \quad y(0) = 1 \quad (3.1.5)$$

at $x = 0.1, 0.2, 0.3$.

Solution We rewrite (3.1.5) as

$$y' = -2y + x^3 e^{-2x}, \quad y(0) = 1,$$

which is of the form (3.1.1), with

$$f(x, y) = -2y + x^3 e^{-2x}, \quad x_0 = 0, \text{ and } y_0 = 1.$$

Euler's method yields

$$\begin{aligned} y_1 &= y_0 + hf(x_0, y_0) \\ &= 1 + (.1)f(0, 1) = 1 + (.1)(-2) = .8, \\ y_2 &= y_1 + hf(x_1, y_1) \\ &= .8 + (.1)f(.1, .8) = .8 + (.1)\left(-2(.8) + (.1)^3 e^{-.2}\right) = .640081873, \\ y_3 &= y_2 + hf(x_2, y_2) \\ &= .640081873 + (.1)\left(-2(.640081873) + (.2)^3 e^{-.4}\right) = .512601754. \blacksquare \end{aligned}$$

Example 3.1.2 Use Euler's method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to find approximate values of the solution of the initial value problem

$$y' + 2y = x^3 e^{-2x}, \quad y(0) = 1$$

at $x = 0, 0.1, 0.2, 0.3, \dots, 1.0$. Compare these approximate values with the values of the exact solution

$$y = \frac{e^{-2x}}{4}(x^4 + 4), \quad (3.1.6)$$

which can be obtained by the method of Section 2.1. (Verify.)

Solution Table 3.1.1 shows the values of the exact solution (3.1.6) at the specified points, and the approximate values of the solution at these points obtained by Euler's method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$. In examining this table, keep in mind that the approximate values in the column corresponding to $h = .05$ are actually the results of 20 steps with Euler's method. We haven't listed the estimates of the solution obtained for $x = 0.05, 0.15, \dots$, since there's nothing to compare them with in the column corresponding to $h = 0.1$. Similarly, the approximate values in the column corresponding to $h = 0.025$ are actually the results of 40 steps with Euler's method.

Table 3.1.1. Numerical solution of $y' + 2y = x^3 e^{-2x}$, $y(0) = 1$, by Euler's method.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	Exact
0.0	1.000000000	1.000000000	1.000000000	1.000000000
0.1	0.800000000	0.810005655	0.814518349	0.818751221
0.2	0.640081873	0.656266437	0.663635953	0.670588174
0.3	0.512601754	0.532290981	0.541339495	0.549922980
0.4	0.411563195	0.432887056	0.442774766	0.452204669
0.5	0.332126261	0.353785015	0.363915597	0.373627557
0.6	0.270299502	0.291404256	0.301359885	0.310952904
0.7	0.222745397	0.242707257	0.252202935	0.261398947
0.8	0.186654593	0.205105754	0.213956311	0.222570721
0.9	0.159660776	0.176396883	0.184492463	0.192412038
1.0	0.139778910	0.154715925	0.162003293	0.169169104

You can see from Table 3.1.1 that decreasing the step size improves the accuracy of Euler's method. For example,

$$y_{\text{exact}}(1) - y_{\text{approx}}(1) \approx \begin{cases} .0293 & \text{with } h = 0.1, \\ .0144 & \text{with } h = 0.05, \\ .0071 & \text{with } h = 0.025. \end{cases}$$

Based on this scanty evidence, you might guess that the error in approximating the exact solution at a *fixed value of* x by Euler's method is roughly halved when the step size is halved. You can find more evidence to support this conjecture by examining Table 3.1.2, which lists the approximate values of $y_{\text{exact}} - y_{\text{approx}}$ at $x = 0.1, 0.2, \dots, 1.0$.

Table 3.1.2. Errors in approximate solutions of $y' + 2y = x^3 e^{-2x}$, $y(0) = 1$, obtained by Euler's method.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$
0.1	0.0187	0.0087	0.0042
0.2	0.0305	0.0143	0.0069
0.3	0.0373	0.0176	0.0085
0.4	0.0406	0.0193	0.0094
0.5	0.0415	0.0198	0.0097
0.6	0.0406	0.0195	0.0095
0.7	0.0386	0.0186	0.0091
0.8	0.0359	0.0174	0.0086
0.9	0.0327	0.0160	0.0079
1.0	0.0293	0.0144	0.0071

Example 3.1.3 Tables 3.1.3 and 3.1.4 show analogous results for the nonlinear initial value problem

$$y' = -2y^2 + xy + x^2, \quad y(0) = 1, \quad (3.1.7)$$

except in this case we can't solve (3.1.7) exactly. The results in the "Exact" column were obtained by using a more accurate numerical method known as the *Runge-Kutta* method with a small step size. They are exact to eight decimal places. ■

Table 3.1.3. Numerical solution of $y' = -2y^2 + xy + x^2$, $y(0) = 1$, by Euler's method.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	"Exact"
0.0	1.000000000	1.000000000	1.000000000	1.000000000
0.1	0.800000000	0.821375000	0.829977007	0.837584494
0.2	0.681000000	0.707795377	0.719226253	0.729641890
0.3	0.605867800	0.633776590	0.646115227	0.657580377
0.4	0.559628676	0.587454526	0.600045701	0.611901791
0.5	0.535376972	0.562906169	0.575556391	0.587575491
0.6	0.529820120	0.557143535	0.569824171	0.581942225
0.7	0.541467455	0.568716935	0.581435423	0.593629526
0.8	0.569732776	0.596951988	0.609684903	0.621907458
0.9	0.614392311	0.641457729	0.654110862	0.666250842
1.0	0.675192037	0.701764495	0.714151626	0.726015790

Table 3.1.4. Errors in approximate solutions of $y' = -2y^2 + xy + x^2$, $y(0) = 1$, obtained by Euler's method.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$
0.1	0.0376	0.0162	0.0076
0.2	0.0486	0.0218	0.0104
0.3	0.0517	0.0238	0.0115
0.4	0.0523	0.0244	0.0119
0.5	0.0522	0.0247	0.0121
0.6	0.0521	0.0248	0.0121
0.7	0.0522	0.0249	0.0122
0.8	0.0522	0.0250	0.0122
0.9	0.0519	0.0248	0.0121
1.0	0.0508	0.0243	0.0119

Example 3.1.4 In Example 2.1.7 we had to leave the solution of the initial value problem

$$y' - 2xy = 1, \quad y(0) = 3 \quad (3.1.22)$$

in the form

$$y = e^{x^2} \left(3 + \int_0^x e^{-t^2} dt \right) \quad (3.1.23)$$

because it was impossible to evaluate this integral exactly in terms of elementary functions. Use step sizes $h = 0.2$, $h = 0.1$, and $h = 0.05$ to find approximate values of the solution of (3.1.22) at $x = 0, 0.2, 0.4, 0.6, \dots, 2.0$ by (a) Euler's method; (b) the Euler semilinear method.

Solution (a) Rewriting (3.1.22) as

$$y' = 1 + 2xy, \quad y(0) = 3 \quad (3.1.24)$$

and applying Euler's method with $f(x, y) = 1 + 2xy$ yields the results shown in Table 3.1.5. Because of the large differences between the estimates obtained for the three values of h , it would be clear that these results are useless even if the “exact” values were not included in the table.

Table 3.1.5. Numerical solution of $y' - 2xy = 1$, $y(0) = 3$, with Euler's method.

x	$h = 0.2$	$h = 0.1$	$h = 0.05$	“Exact”
0.0	3.000000000	3.000000000	3.000000000	3.000000000
0.2	3.200000000	3.262000000	3.294348537	3.327851973
0.4	3.656000000	3.802028800	3.881421103	3.966059348
0.6	4.440960000	4.726810214	4.888870783	5.067039535
0.8	5.706790400	6.249191282	6.570796235	6.936700945
1.0	7.732963328	8.771893026	9.419105620	10.184923955
1.2	11.026148659	13.064051391	14.405772067	16.067111677
1.4	16.518700016	20.637273893	23.522935872	27.289392347
1.6	25.969172024	34.570423758	41.033441257	50.000377775
1.8	42.789442120	61.382165543	76.491018246	98.982969504
2.0	73.797840446	115.440048291	152.363866569	211.954462214

It's easy to see why Euler's method yields such poor results. Recall that the constant M in (3.1.10) – which plays an important role in determining the local truncation error in Euler's method – must be an upper bound for the values of the second derivative y'' of the solution of the initial value problem (3.1.22) on $(0, 2)$. The problem is that y'' assumes very large values on this interval. To see this, we differentiate (3.1.24) to obtain

$$y''(x) = 2y(x) + 2xy'(x) = 2y(x) + 2x(1 + 2xy(x)) = 2(1 + 2x^2)y(x) + 2x,$$

where the second equality follows again from (3.1.24). Since (3.1.23) implies that $y(x) > 3e^{x^2}$ if $x > 0$,

$$y''(x) > 6(1 + 2x^2)e^{x^2} + 2x, \quad x > 0.$$

For example, letting $x = 2$ shows that $y''(2) > 2952$.

Solution (b) Since $y_1 = e^{x^2}$ is a solution of the complementary equation $y' - 2xy = 0$, we can apply the Euler semilinear method to (3.1.22), with

$$y = ue^{x^2} \quad \text{and} \quad u' = e^{-x^2}, \quad u(0) = 3.$$

The results listed in Table 3.1.6 are clearly better than those obtained by Euler's method.

Table 3.1.6. Numerical solution of $y' - 2xy = 1$, $y(0) = 3$, by the Euler semilinear method.

x	$h = 0.2$	$h = 0.1$	$h = 0.05$	“Exact”
0.0	3.000000000	3.000000000	3.000000000	3.000000000
0.2	3.330594477	3.329558853	3.328788889	3.327851973
0.4	3.980734157	3.974067628	3.970230415	3.966059348
0.6	5.106360231	5.087705244	5.077622723	5.067039535
0.8	7.021003417	6.980190891	6.958779586	6.936700945
1.0	10.350076600	10.269170824	10.227464299	10.184923955
1.2	16.381180092	16.226146390	16.147129067	16.067111677
1.4	27.890003380	27.592026085	27.441292235	27.289392347
1.6	51.183323262	50.594503863	50.298106659	50.000377775
1.8	101.424397595	100.206659076	99.595562766	98.982969504
2.0	217.301032800	214.631041938	213.293582978	211.954462214

Example 3.1.5 Applying Euler’s method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to the initial value problem

$$y' - 2y = \frac{x}{1 + y^2}, \quad y(1) = 7 \quad (3.1.25)$$

on $[1, 2]$ yields the results in Table 3.1.7. Applying the Euler semilinear method with

$$y = ue^{2x} \quad \text{and} \quad u' = \frac{xe^{-2x}}{1 + u^2e^{4x}}, \quad u(1) = 7e^{-2}$$

yields the results in Table 3.1.8. Since the latter are clearly less dependent on step size than the former, we conclude that the Euler semilinear method is better than Euler’s method for (3.1.25). This conclusion is supported by comparing the approximate results obtained by the two methods with the “exact” values of the solution.

Table 3.1.7. Numerical solution of $y' - 2y = x/(1 + y^2)$, $y(1) = 7$, by Euler’s method.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	“Exact”
1.0	7.000000000	7.000000000	7.000000000	7.000000000
1.1	8.402000000	8.471970569	8.510493955	8.551744786
1.2	10.083936450	10.252570169	10.346014101	10.446546230
1.3	12.101892354	12.406719381	12.576720827	12.760480158
1.4	14.523152445	15.012952416	15.287872104	15.586440425
1.5	17.428443554	18.166277405	18.583079406	19.037865752
1.6	20.914624471	21.981638487	22.588266217	23.253292359
1.7	25.097914310	26.598105180	27.456479695	28.401914416
1.8	30.117766627	32.183941340	33.373738944	34.690375086
1.9	36.141518172	38.942738252	40.566143158	42.371060528
2.0	43.369967155	47.120835251	49.308511126	51.752229656

Table 3.1.8. Numerical solution of $y' - 2y = x/(1 + y^2)$, $y(1) = 7$, by the Euler semilinear method.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	“Exact”
1.0	7.000000000	7.000000000	7.000000000	7.000000000
1.1	8.552262113	8.551993978	8.551867007	8.551744786
1.2	10.447568674	10.447038547	10.446787646	10.446546230
1.3	12.762019799	12.761221313	12.760843543	12.760480158
1.4	15.588535141	15.587448600	15.586934680	15.586440425
1.5	19.040580614	19.039172241	19.038506211	19.037865752
1.6	23.256721636	23.254942517	23.254101253	23.253292359
1.7	28.406184597	28.403969107	28.402921581	28.401914416
1.8	34.695649222	34.692912768	34.691618979	34.690375086
1.9	42.377544138	42.374180090	42.372589624	42.371060528
2.0	51.760178446	51.756054133	51.754104262	51.752229656

Example 3.1.6 Applying Euler’s method with step sizes $h = 0.1$, $h = 0.05$, and $h = 0.025$ to the initial value problem

$$y' + 3x^2y = 1 + y^2, \quad y(2) = 2 \quad (3.1.26)$$

on $[2, 3]$ yields the results in Table 3.1.9. Applying the Euler semilinear method with

$$y = ue^{-x^3} \quad \text{and} \quad u' = e^{x^3}(1 + u^2e^{-2x^3}), \quad u(2) = 2e^8$$

yields the results in Table 3.1.10. Noting the close agreement among the three columns of Table 3.1.9 (at least for larger values of x) and the lack of any such agreement among the columns of Table 3.1.10, we conclude that Euler’s method is better than the Euler semilinear method for (3.1.26). Comparing the results with the exact values supports this conclusion.

Table 3.1.9. Numerical solution of $y' + 3x^2y = 1 + y^2$, $y(2) = 2$, by Euler’s method.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	“Exact”
2.0	2.000000000	2.000000000	2.000000000	2.000000000
2.1	0.100000000	0.493231250	0.609611171	0.701162906
2.2	0.068700000	0.122879586	0.180113445	0.236986800
2.3	0.069419569	0.070670890	0.083934459	0.103815729
2.4	0.059732621	0.061338956	0.063337561	0.068390786
2.5	0.056871451	0.056002363	0.056249670	0.057281091
2.6	0.050560917	0.051465256	0.051517501	0.051711676
2.7	0.048279018	0.047484716	0.047514202	0.047564141
2.8	0.042925892	0.043967002	0.043989239	0.044014438
2.9	0.042148458	0.040839683	0.040857109	0.040875333
3.0	0.035985548	0.038044692	0.038058536	0.038072838

Table 3.1.10. Numerical solution of $y' + 3x^2y = 1 + y^2$, $y(2) = 2$, by the Euler semilinear method.

x	$h = 0.1$	$h = 0.05$	$h = 0.025$	“Exact”
x	$h = 0.1$	$h = 0.05$	$h = 0.025$	$h = .0125$
2.0	2.000000000	2.000000000	2.000000000	2.000000000
2.1	0.708426286	0.702568171	0.701214274	0.701162906
2.2	0.214501852	0.222599468	0.228942240	0.236986800
2.3	0.069861436	0.083620494	0.092852806	0.103815729
2.4	0.032487396	0.047079261	0.056825805	0.068390786
2.5	0.021895559	0.036030018	0.045683801	0.057281091
2.6	0.017332058	0.030750181	0.040189920	0.051711676
2.7	0.014271492	0.026931911	0.036134674	0.047564141
2.8	0.011819555	0.023720670	0.032679767	0.044014438
2.9	0.009776792	0.020925522	0.029636506	0.040875333
3.0	0.008065020	0.018472302	0.026931099	0.038072838

Section 3.2

Example 3.2.1 Use the improved Euler method with $h = 0.1$ to find approximate values of the solution of the initial value problem

$$y' + 2y = x^3 e^{-2x}, \quad y(0) = 1 \quad (3.2.5)$$

at $x = 0.1, 0.2, 0.3$.

Solution As in Example 3.1.1, we rewrite (3.2.5) as

$$y' = -2y + x^3 e^{-2x}, \quad y(0) = 1,$$

which is of the form (3.2.1), with

$$f(x, y) = -2y + x^3 e^{-2x}, \quad x_0 = 0, \quad \text{and } y_0 = 1.$$

The improved Euler method yields

$$\begin{aligned}
k_{10} &= f(x_0, y_0) = f(0, 1) = -2, \\
k_{20} &= f(x_1, y_0 + hk_{10}) = f(.1, 1 + (.1)(-2)) \\
&= f(.1, .8) = -2(.8) + (.1)^3 e^{-.2} = -1.599181269, \\
y_1 &= y_0 + \frac{h}{2}(k_{10} + k_{20}), \\
&= 1 + (.05)(-2 - 1.599181269) = .820040937, \\
k_{11} &= f(x_1, y_1) = f(.1, .820040937) = -2(.820040937) + (.1)^3 e^{-.2} = -1.639263142, \\
k_{21} &= f(x_2, y_1 + hk_{11}) = f(.2, .820040937 + .1(-1.639263142)), \\
&= f(.2, .656114622) = -2(.656114622) + (.2)^3 e^{-.4} = -1.306866684, \\
y_2 &= y_1 + \frac{h}{2}(k_{11} + k_{21}), \\
&= .820040937 + (.05)(-1.639263142 - 1.306866684) = .672734445, \\
k_{12} &= f(x_2, y_2) = f(.2, .672734445) = -2(.672734445) + (.2)^3 e^{-.4} = -1.340106330, \\
k_{22} &= f(x_3, y_2 + hk_{12}) = f(.3, .672734445 + .1(-1.340106330)), \\
&= f(.3, .538723812) = -2(.538723812) + (.3)^3 e^{-.6} = -1.062629710, \\
y_3 &= y_2 + \frac{h}{2}(k_{12} + k_{22}) \\
&= .672734445 + (.05)(-1.340106330 - 1.062629710) = .552597643.
\end{aligned}$$

Example 3.2.2 Table 3.2.2 shows results of using the improved Euler method with step sizes $h = 0.1$ and $h = 0.05$ to find approximate values of the solution of the initial value problem

$$y' + 2y = x^3 e^{-2x}, \quad y(0) = 1$$

at $x = 0, 0.1, 0.2, 0.3, \dots, 1.0$. For comparison, it also shows the corresponding approximate values obtained with Euler's method in 3.1.2, and the values of the exact solution

$$y = \frac{e^{-2x}}{4}(x^4 + 4).$$

The results obtained by the improved Euler method with $h = 0.1$ are better than those obtained by Euler's method with $h = 0.05$.

Table 3.2.2. Numerical solution of $y' + 2y = x^3 e^{-2x}$, $y(0) = 1$, by Euler's method and the improved Euler method.

x	$h = 0.1$	$h = 0.05$	$h = 0.1$	$h = 0.05$	Exact
0.0	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
0.1	0.800000000	0.810005655	0.820040937	0.819050572	0.818751221
0.2	0.640081873	0.656266437	0.672734445	0.671086455	0.670588174
0.3	0.512601754	0.532290981	0.552597643	0.550543878	0.549922980
0.4	0.411563195	0.432887056	0.455160637	0.452890616	0.452204669
0.5	0.332126261	0.353785015	0.376681251	0.374335747	0.373627557
0.6	0.270299502	0.291404256	0.313970920	0.311652239	0.310952904
0.7	0.222745397	0.242707257	0.264287611	0.262067624	0.261398947
0.8	0.186654593	0.205105754	0.225267702	0.223194281	0.222570721
0.9	0.159660776	0.176396883	0.194879501	0.192981757	0.192412038
1.0	0.139778910	0.154715925	0.171388070	0.169680673	0.169169104
	Euler		Improved Euler		Exact

Example 3.2.3 Table 3.2.3 shows analogous results for the nonlinear initial value problem

$$y' = -2y^2 + xy + x^2, \quad y(0) = 1.$$

We applied Euler's method to this problem in Example 3.1.3.

Table 3.2.3. Numerical solution of $y' = -2y^2 + xy + x^2$, $y(0) = 1$, by Euler's method and the improved Euler method.

x	$h = 0.1$	$h = 0.05$	$h = 0.1$	$h = 0.05$	"Exact"
0.0	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
0.1	0.800000000	0.821375000	0.840500000	0.838288371	0.837584494
0.2	0.681000000	0.707795377	0.733430846	0.730556677	0.729641890
0.3	0.605867800	0.633776590	0.661600806	0.658552190	0.657580377
0.4	0.559628676	0.587454526	0.615961841	0.612884493	0.611901791
0.5	0.535376972	0.562906169	0.591634742	0.588558952	0.587575491
0.6	0.529820120	0.557143535	0.586006935	0.582927224	0.581942225
0.7	0.541467455	0.568716935	0.597712120	0.594618012	0.593629526
0.8	0.569732776	0.596951988	0.626008824	0.622898279	0.621907458
0.9	0.614392311	0.641457729	0.670351225	0.667237617	0.666250842
1.0	0.675192037	0.701764495	0.730069610	0.726985837	0.726015790
	Euler		Improved Euler		"Exact"

Example 3.2.4 Use step sizes $h = 0.2$, $h = 0.1$, and $h = 0.05$ to find approximate values of the solution of

$$y' - 2xy = 1, \quad y(0) = 3 \quad (3.2.6)$$

at $x = 0, 0.2, 0.4, 0.6, \dots, 2.0$ by (a) the improved Euler method; (b) the improved Euler semilinear method. (We used Euler's method and the Euler semilinear method on this problem in 3.1.4.)

Solution (a) Rewriting (3.2.6) as

$$y' = 1 + 2xy, \quad y(0) = 3$$

and applying the improved Euler method with $f(x, y) = 1 + 2xy$ yields the results shown in Table 3.2.4.

Solution (b) Since $y_1 = e^{x^2}$ is a solution of the complementary equation $y' - 2xy = 0$, we can apply the improved Euler semilinear method to (3.2.6), with

$$y = ue^{x^2} \quad \text{and} \quad u' = e^{-x^2}, \quad u(0) = 3.$$

The results listed in Table 3.2.5 are clearly better than those obtained by the improved Euler method.

Table 3.2.4. Numerical solution of $y' - 2xy = 1$, $y(0) = 3$, by the improved Euler method.

x	$h = 0.2$	$h = 0.1$	$h = 0.05$	“Exact”
0.0	3.000000000	3.000000000	3.000000000	3.000000000
0.2	3.328000000	3.328182400	3.327973600	3.327851973
0.4	3.964659200	3.966340117	3.966216690	3.966059348
0.6	5.057712497	5.065700515	5.066848381	5.067039535
0.8	6.900088156	6.928648973	6.934862367	6.936700945
1.0	10.065725534	10.154872547	10.177430736	10.184923955
1.2	15.708954420	15.970033261	16.041904862	16.067111677
1.4	26.244894192	26.991620960	27.210001715	27.289392347
1.6	46.958915746	49.096125524	49.754131060	50.000377775
1.8	89.982312641	96.200506218	98.210577385	98.982969504
2.0	184.563776288	203.151922739	209.464744495	211.954462214

Table 3.2.5. Numerical solution of $y' - 2xy = 1$, $y(0) = 3$, by the improved Euler semilinear method.

x	$h = 0.2$	$h = 0.1$	$h = 0.05$	“Exact”
0.0	3.000000000	3.000000000	3.000000000	3.000000000
0.2	3.326513400	3.327518315	3.327768620	3.327851973
0.4	3.963383070	3.965392084	3.965892644	3.966059348
0.6	5.063027290	5.066038774	5.066789487	5.067039535
0.8	6.931355329	6.935366847	6.936367564	6.936700945
1.0	10.178248417	10.183256733	10.184507253	10.184923955
1.2	16.059110511	16.065111599	16.066611672	16.067111677
1.4	27.280070674	27.287059732	27.288809058	27.289392347
1.6	49.989741531	49.997712997	49.999711226	50.000377775
1.8	98.971025420	98.979972988	98.982219722	98.982969504
2.0	211.941217796	211.951134436	211.953629228	211.954462214

Section 3.3

Example 3.3.1 Use the Runge-Kutta method with $h = 0.1$ to find approximate values for the solution of the initial value problem

$$y' + 2y = x^3 e^{-2x}, \quad y(0) = 1, \quad (3.3.2)$$

at $x = 0.1, 0.2$.

Solution Again we rewrite (3.3.2) as

$$y' = -2y + x^3 e^{-2x}, \quad y(0) = 1,$$

which is of the form (3.3.1), with

$$f(x, y) = -2y + x^3 e^{-2x}, \quad x_0 = 0, \text{ and } y_0 = 1.$$

The Runge-Kutta method yields

$$\begin{aligned} k_{10} &= f(x_0, y_0) = f(0, 1) = -2, \\ k_{20} &= f(x_0 + h/2, y_0 + hk_{10}/2) = f(.05, 1 + (.05)(-2)) \\ &= f(.05, .9) = -2(.9) + (.05)^3 e^{-.1} = -1.799886895, \\ k_{30} &= f(x_0 + h/2, y_0 + hk_{20}/2) = f(.05, 1 + (.05)(-1.799886895)) \\ &= f(.05, .910005655) = -2(.910005655) + (.05)^3 e^{-.1} = -1.819898206, \\ k_{40} &= f(x_0 + h, y_0 + hk_{30}) = f(.1, 1 + (.1)(-1.819898206)) \\ &= f(.1, .818010179) = -2(.818010179) + (.1)^3 e^{-.2} = -1.635201628, \\ y_1 &= y_0 + \frac{h}{6}(k_{10} + 2k_{20} + 2k_{30} + k_{40}), \\ &= 1 + \frac{.1}{6}(-2 + 2(-1.799886895) + 2(-1.819898206) - 1.635201628) = .818753803, \\ k_{11} &= f(x_1, y_1) = f(.1, .818753803) = -2(.818753803) + (.1)^3 e^{-.2} = -1.636688875, \\ k_{21} &= f(x_1 + h/2, y_1 + hk_{11}/2) = f(.15, .818753803 + (.05)(-1.636688875)) \\ &= f(.15, .736919359) = -2(.736919359) + (.15)^3 e^{-.3} = -1.471338457, \\ k_{31} &= f(x_1 + h/2, y_1 + hk_{21}/2) = f(.15, .818753803 + (.05)(-1.471338457)) \\ &= f(.15, .745186880) = -2(.745186880) + (.15)^3 e^{-.3} = -1.487873498, \\ k_{41} &= f(x_1 + h, y_1 + hk_{31}) = f(.2, .818753803 + (.1)(-1.487873498)) \\ &= f(.2, .669966453) = -2(.669966453) + (.2)^3 e^{-.4} = -1.334570346, \\ y_2 &= y_1 + \frac{h}{6}(k_{11} + 2k_{21} + 2k_{31} + k_{41}), \\ &= .818753803 + \frac{.1}{6}(-1.636688875 + 2(-1.471338457) + 2(-1.487873498) - 1.334570346) \\ &= .670592417. \end{aligned}$$

Example 3.3.2 Table 3.3.1 shows results of using the Runge-Kutta method with step sizes $h = 0.1$ and $h = 0.05$ to find approximate values of the solution of the initial value problem

$$y' + 2y = x^3 e^{-2x}, \quad y(0) = 1$$

at $x = 0, 0.1, 0.2, 0.3, \dots, 1.0$. For comparison, it also shows the corresponding approximate values obtained with the improved Euler method in Example 3.2.2, and the values of the exact solution

$$y = \frac{e^{-2x}}{4}(x^4 + 4).$$

The results obtained by the Runge-Kutta method are clearly better than those obtained by the improved Euler method; in fact, the results obtained by the Runge-Kutta method with $h = 0.1$ are better than those obtained by the improved Euler method with $h = 0.05$.

Table 3.3.1. Numerical solution of $y' + 2y = x^3 e^{-2x}$, $y(0) = 1$, by the Runge-Kutta method and the improved Euler method.

x	$h = 0.1$	$h = 0.05$	$h = 0.1$	$h = 0.05$	Exact
0.0	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
0.1	0.820040937	0.819050572	0.818753803	0.818751370	0.818751221
0.2	0.672734445	0.671086455	0.670592417	0.670588418	0.670588174
0.3	0.552597643	0.550543878	0.549928221	0.549923281	0.549922980
0.4	0.455160637	0.452890616	0.452210430	0.452205001	0.452204669
0.5	0.376681251	0.374335747	0.373633492	0.373627899	0.373627557
0.6	0.313970920	0.311652239	0.310958768	0.310953242	0.310952904
0.7	0.264287611	0.262067624	0.261404568	0.261399270	0.261398947
0.8	0.225267702	0.223194281	0.222575989	0.222571024	0.222570721
0.9	0.194879501	0.192981757	0.192416882	0.192412317	0.192412038
1.0	0.171388070	0.169680673	0.169173489	0.169169356	0.169169104
	Improved Euler		Runge-Kutta		Exact

Example 3.3.3 Table 3.3.2 shows analogous results for the nonlinear initial value problem

$$y' = -2y^2 + xy + x^2, \quad y(0) = 1.$$

We applied the improved Euler method to this problem in Example 3.2.3.

Table 3.3.2. Numerical solution of $y' = -2y^2 + xy + x^2$, $y(0) = 1$, by the Runge-Kutta method and the improved Euler method.

x	$h = 0.1$	$h = 0.05$	$h = 0.1$	$h = 0.05$	“Exact”
0.0	1.000000000	1.000000000	1.000000000	1.000000000	1.000000000
0.1	0.840500000	0.838288371	0.837587192	0.837584759	0.837584494
0.2	0.733430846	0.730556677	0.729644487	0.729642155	0.729641890
0.3	0.661600806	0.658552190	0.657582449	0.657580598	0.657580377
0.4	0.615961841	0.612884493	0.611903380	0.611901969	0.611901791
0.5	0.591634742	0.588558952	0.587576716	0.587575635	0.587575491
0.6	0.586006935	0.582927224	0.581943210	0.581942342	0.581942225
0.7	0.597712120	0.594618012	0.593630403	0.593629627	0.593629526
0.8	0.626008824	0.622898279	0.621908378	0.621907553	0.621907458
0.9	0.670351225	0.667237617	0.666251988	0.666250942	0.666250842
1.0	0.730069610	0.726985837	0.726017378	0.726015908	0.726015790
	Improved Euler		Runge-Kutta		“Exact”

Example 3.3.4 Tables 3.3.3 and 3.3.4 show results obtained by applying the Runge-Kutta and Runge-Kutta semilinear methods to the initial value problem

$$y' - 2xy = 1, \quad y(0) = 3,$$

which we considered in Examples 3.1.4 and 3.2.4.

Table 3.3.3. Numerical solution of $y' - 2xy = 1$, $y(0) = 3$, by the Runge-Kutta method.

x	$h = 0.2$	$h = 0.1$	$h = 0.05$	“Exact”
0.0	3.000000000	3.000000000	3.000000000	3.000000000
0.2	3.327846400	3.327851633	3.327851952	3.327851973
0.4	3.966044973	3.966058535	3.966059300	3.966059348
0.6	5.066996754	5.067037123	5.067039396	5.067039535
0.8	6.936534178	6.936690679	6.936700320	6.936700945
1.0	10.184232252	10.184877733	10.184920997	10.184923955
1.2	16.064344805	16.066915583	16.067098699	16.067111677
1.4	27.278771833	27.288605217	27.289338955	27.289392347
1.6	49.960553660	49.997313966	50.000165744	50.000377775
1.8	98.834337815	98.971146146	98.982136702	98.982969504
2.0	211.393800152	211.908445283	211.951167637	211.954462214

Table 3.3.4. Numerical solution of $y' - 2xy = 1$, $y(0) = 3$, by the Runge-Kutta semilinear method.

x	$h = 0.2$	$h = 0.1$	$h = 0.05$	“Exact”
0.0	3.000000000	3.000000000	3.000000000	3.000000000
0.2	3.327853286	3.327852055	3.327851978	3.327851973
0.4	3.966061755	3.966059497	3.966059357	3.966059348
0.6	5.067042602	5.067039725	5.067039547	5.067039535
0.8	6.936704019	6.936701137	6.936700957	6.936700945
1.0	10.184926171	10.184924093	10.184923963	10.184923955
1.2	16.067111961	16.067111696	16.067111678	16.067111677
1.4	27.289389418	27.289392167	27.289392335	27.289392347
1.6	50.000370152	50.000377302	50.000377745	50.000377775
1.8	98.982955511	98.982968633	98.982969450	98.982969504
2.0	211.954439983	211.954460825	211.954462127	211.954462214

Example 3.3.5 Use the Runge-Kutta method with step size $h = 0.1$ to find approximate values of the solution of

$$(y - 1)^2 y' = 2x + 3, \quad y(1) = 4 \quad (3.3.5)$$

at $x = 0, 0.1, 0.2, \dots, 1$.

Solution We first rewrite (3.3.5) in the form (3.3.3) as

$$y' = \frac{2x + 3}{(y - 1)^2}, \quad y(1) = 4. \quad (3.3.6)$$

Since the initial condition $y(1) = 4$ is imposed at the right endpoint of the interval $[0, 1]$, we apply the Runge-Kutta method to the initial value problem

$$z' = \frac{2x - 3}{(z - 1)^2}, \quad z(-1) = 4 \quad (3.3.7)$$

on the interval $[-1, 0]$. (You should verify that (3.3.7) is related to (3.3.6) as (3.3.4) is related to (3.3.3).) Table 3.3.5 shows the results. Reversing the order of the rows in Table 3.3.5 and changing the signs of the values of x yields the first two columns of Table 3.3.6. The last column of Table 3.3.6 shows the exact values of y , which are given by

$$y = 1 + (3x^2 + 9x + 15)^{1/3}.$$

(Since the differential equation in (3.3.6) is separable, this formula can be obtained by the method of Section 2.2.)

Table 3.3.5. Numerical solution of $z' = \frac{2x-3}{(z-1)^2}$, $z(-1) = 4$, on $[-1, 0]$.

x	z
-1.0	4.000000000
-0.9	3.944536474
-0.8	3.889298649
-0.7	3.834355648
-0.6	3.779786399
-0.5	3.725680888
-0.4	3.672141529
-0.3	3.619284615
-0.2	3.567241862
-0.1	3.516161955
0.0	3.466212070

Table 3.3.6. Numerical solution of $(y-1)^2 y' = 2x+3$, $y(1) = 4$, on $[0, 1]$.

x	y	Exact
0.00	3.466212070	3.466212074
0.10	3.516161955	3.516161958
0.20	3.567241862	3.567241864
0.30	3.619284615	3.619284617
0.40	3.672141529	3.672141530
0.50	3.725680888	3.725680889
0.60	3.779786399	3.779786399
0.70	3.834355648	3.834355648
0.80	3.889298649	3.889298649
0.90	3.944536474	3.944536474
1.00	4.000000000	4.000000000

Section 4.1

Example 4.1.1 A radioactive substance has a half-life of 1620 years.

- (a) If its mass is now 4 g (grams), how much will be left 810 years from now?
 (b) Find the time t_1 when 1.5 g of the substance remain.

Solution (a) From (4.1.3) with $t_0 = 0$ and $Q_0 = 4$,

$$Q = 4e^{-kt}, \quad (4.1.6)$$

where we determine k from (4.1.5), with $\tau = 1620$ years:

$$k = \frac{\ln 2}{\tau} = \frac{\ln 2}{1620}.$$

Substituting this in (4.1.6) yields

$$Q = 4e^{-(t \ln 2)/1620}. \quad (4.1.7)$$

Therefore the mass left after 810 years will be

$$\begin{aligned} Q(810) &= 4e^{-(810 \ln 2)/1620} = 4e^{-(\ln 2)/2} \\ &= 2\sqrt{2} \text{ g.} \end{aligned}$$

Solution (b) Setting $t = t_1$ in (4.1.7) and requiring that $Q(t_1) = 1.5$ yields

$$\frac{3}{2} = 4e^{(-t_1 \ln 2)/1620}.$$

Dividing by 4 and taking logarithms yields

$$\ln \frac{3}{8} = -\frac{t_1 \ln 2}{1620}.$$

Since $\ln 3/8 = -\ln 8/3$,

$$t_1 = 1620 \frac{\ln 8/3}{\ln 2} \approx 2292.4 \text{ years.}$$

Example 4.1.2 If \$150 is deposited in a bank that pays $5\frac{1}{2}\%$ annual interest compounded continuously, the value of the account after t years is

$$Q(t) = 150e^{.055t}$$

dollars. (Note that it's necessary to write the interest rate as a decimal; thus, $r = .055$.) Therefore, after $t = 10$ years the value of the account is

$$Q(10) = 150e^{.55} \approx \$259.99.$$

Example 4.1.3 We wish to accumulate \$10,000 in 10 years by making a single deposit in a savings account bearing $5\frac{1}{2}\%$ annual interest compounded continuously. How much must we deposit in the account?

Solution The value of the account at time t is

$$Q(t) = Q_0 e^{.055t}. \quad (4.1.9)$$

Since we want $Q(10)$ to be \$10,000, the initial deposit Q_0 must satisfy the equation

$$10000 = Q_0 e^{.55}, \quad (4.1.10)$$

obtained by setting $t = 10$ and $Q(10) = 10000$ in (4.1.9). Solving (4.1.10) for Q_0 yields

$$Q_0 = 10000e^{-.55} \approx \$5769.50.$$

Example 4.1.4 A radioactive substance with decay constant k is produced at a constant rate of a units of mass per unit time.

(a) Assuming that $Q(0) = Q_0$, find the mass $Q(t)$ of the substance present at time t .

(b) Find $\lim_{t \rightarrow \infty} Q(t)$.

Solution (a) Here

$$Q' = \text{rate of increase of } Q - \text{rate of decrease of } Q.$$

The rate of increase is the constant a . Since Q is radioactive with decay constant k , the rate of decrease is kQ . Therefore

$$Q' = a - kQ.$$

This is a linear first order differential equation. Rewriting it and imposing the initial condition shows that Q is the solution of the initial value problem

$$Q' + kQ = a, \quad Q(0) = Q_0. \quad (4.1.11)$$

Since e^{-kt} is a solution of the complementary equation, the solutions of (4.1.11) are of the form $Q = ue^{-kt}$, where $u'e^{-kt} = a$, so $u' = ae^{kt}$. Hence,

$$u = \frac{a}{k}e^{kt} + c$$

and

$$Q = ue^{-kt} = \frac{a}{k} + ce^{-kt}.$$

Since $Q(0) = Q_0$, setting $t = 0$ here yields

$$Q_0 = \frac{a}{k} + c \quad \text{or} \quad c = Q_0 - \frac{a}{k}.$$

Therefore

$$Q = \frac{a}{k} + \left(Q_0 - \frac{a}{k}\right)e^{-kt}. \quad (4.1.12)$$

Solution (b) Since $k > 0$, $\lim_{t \rightarrow \infty} e^{-kt} = 0$, so from (4.1.12)

$$\lim_{t \rightarrow \infty} Q(t) = \frac{a}{k}.$$

This limit depends only on a and k , and not on Q_0 . We say that a/k is the *steady state* value of Q . From (4.1.12) we also see that Q approaches its steady state value from above if $Q_0 > a/k$, or from below if $Q_0 < a/k$. If $Q_0 = a/k$, then Q remains constant (Figure 4.1.3).

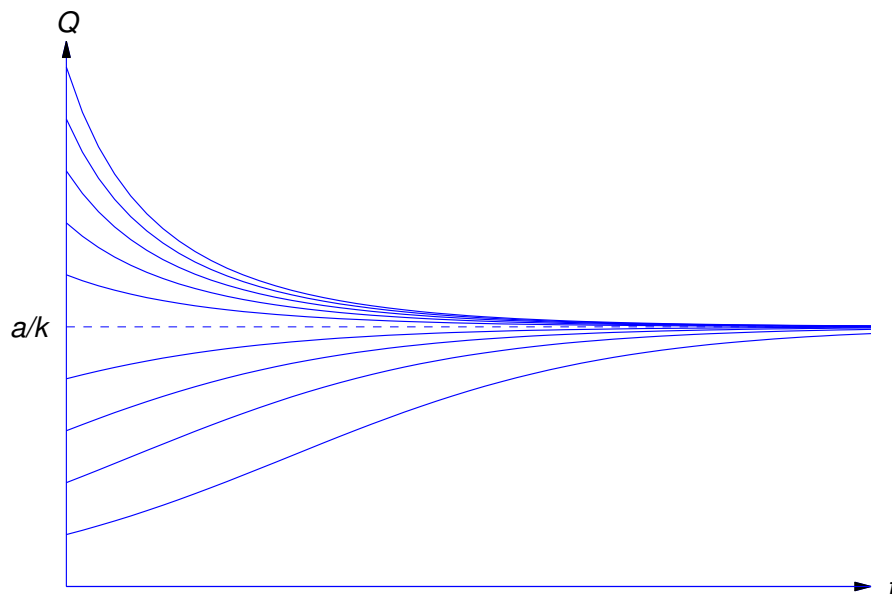


Figure 4.1.3 $Q(t)$ approaches the steady state value $\frac{a}{k}$ as $t \rightarrow \infty$

Example 4.1.5 An archaeologist investigating the site of an ancient village finds a burial ground where the amount of carbon-14 present in individual remains is between 42 and 44% of the amount present in live individuals. Estimate the age of the village and the length of time for which it survived.

Solution Let $Q = Q(t)$ be the quantity of carbon-14 in an individual set of remains t years after death, and let Q_0 be the quantity that would be present in live individuals. Since carbon-14 decays exponentially with half-life 5570 years, its decay constant is

$$k = \frac{\ln 2}{5570}.$$

Therefore

$$Q = Q_0 e^{-t(\ln 2)/5570}$$

if we choose our time scale so that $t_0 = 0$ is the time of death. If we know the present value of Q we can solve this equation for t , the number of years since death occurred. This yields

$$t = -5570 \frac{\ln Q/Q_0}{\ln 2}.$$

It is given that $Q = .42Q_0$ in the remains of individuals who died first. Therefore these deaths occurred about

$$t_1 = -5570 \frac{\ln .42}{\ln 2} \approx 6971$$

years ago. For the most recent deaths, $Q = .44Q_0$; hence, these deaths occurred about

$$t_2 = -5570 \frac{\ln .44}{\ln 2} \approx 6597$$

years ago. Therefore it's reasonable to conclude that the village was founded about 7000 years ago, and lasted for about 400 years.

Example 4.1.6 A person opens a savings account with an initial deposit of \$1000 and subsequently deposits \$50 per week. Find the value $Q(t)$ of the account at time $t > 0$, assuming that the bank pays 6% interest compounded continuously.

Solution Observe that Q isn't continuous, since there are 52 discrete deposits per year of \$50 each. To construct a mathematical model for this problem in the form of a differential equation, we make the simplifying assumption that the deposits are made continuously at a rate of \$2600 per year. This is essential, since solutions of differential equations are continuous functions. With this assumption, Q increases continuously at the rate

$$Q' = 2600 + .06Q$$

and therefore Q satisfies the differential equation

$$Q' - .06Q = 2600. \quad (4.1.13)$$

(Of course, we must recognize that the solution of this equation is an approximation to the true value of Q at any given time. We'll discuss this further below.) Since $e^{.06t}$ is a solution of the complementary equation, the solutions of (4.1.13) are of the form $Q = ue^{.06t}$, where $u'e^{.06t} = 2600$. Hence, $u' = 2600e^{-.06t}$,

$$u = -\frac{2600}{.06}e^{-.06t} + c$$

and

$$Q = ue^{.06t} = -\frac{2600}{.06} + ce^{.06t}. \quad (4.1.14)$$

Setting $t = 0$ and $Q = 1000$ here yields

$$c = 1000 + \frac{2600}{.06},$$

and substituting this into (4.1.14) yields

$$Q = 1000e^{.06t} + \frac{2600}{.06}(e^{.06t} - 1), \quad (4.1.15)$$

where the first term is the value due to the initial deposit and the second is due to the subsequent weekly deposits. ■

Section 4.2

Example 4.2.1 A ceramic insulator is baked at 400°C and cooled in a room in which the temperature is 25°C. After 4 minutes the temperature of the insulator is 200°C. What is its temperature after 8 minutes?

Solution Here $T_0 = 400$ and $T_m = 25$, so (4.2.2) becomes

$$T = 25 + 375e^{-kt}. \quad (4.2.3)$$

We determine k from the stated condition that $T(4) = 200$; that is,

$$200 = 25 + 375e^{-4k};$$

hence,

$$e^{-4k} = \frac{175}{375} = \frac{7}{15}.$$

Taking logarithms and solving for k yields

$$k = -\frac{1}{4} \ln \frac{7}{15} = \frac{1}{4} \ln \frac{15}{7}.$$

Substituting this into (4.2.3) yields

$$T = 25 + 375e^{-\frac{t}{4} \ln \frac{15}{7}}$$

(Figure 4.2.1). Therefore the temperature of the insulator after 8 minutes is

$$\begin{aligned} T(8) &= 25 + 375e^{-2 \ln \frac{15}{7}} \\ &= 25 + 375 \left(\frac{7}{15} \right)^2 \approx 107^\circ\text{C}. \end{aligned}$$

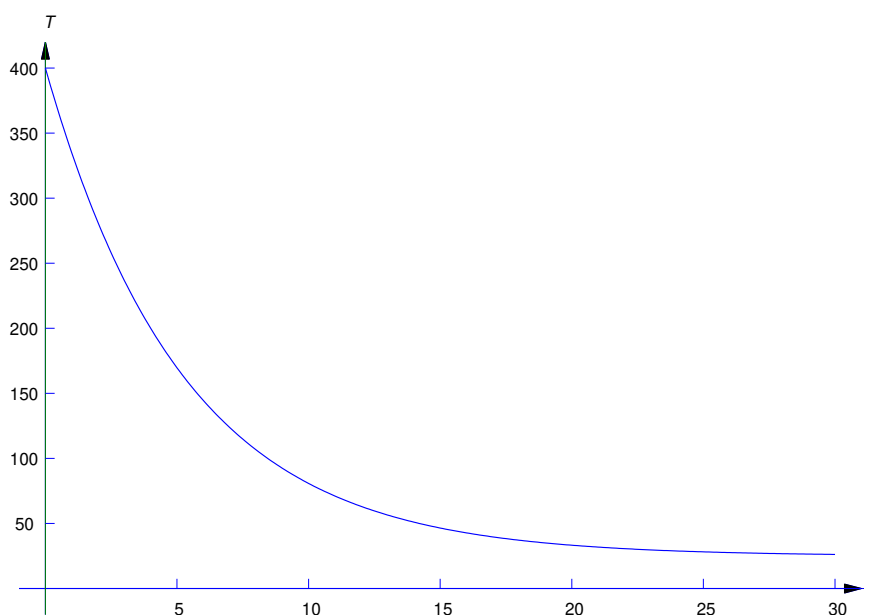


Figure 4.2.1 $T = 25 + 375e^{-(t/4) \ln(15/7)}$

Example 4.2.2 An object with temperature 72°F is placed outside, where the temperature is -20°F . At 11:05 the temperature of the object is 60°F and at 11:07 its temperature is 50°F . At what time was the object placed outside?

Solution Let $T(t)$ be the temperature of the object at time t . For convenience, we choose the origin $t_0 = 0$ of the time scale to be 11:05 so that $T_0 = 60$. We must determine the time τ when $T(\tau) = 72$. Substituting $T_0 = 60$ and $T_m = -20$ into (4.2.2) yields

$$T = -20 + (60 - (-20))e^{-kt}$$

or

$$T = -20 + 80e^{-kt}. \quad (4.2.4)$$

We obtain k from the stated condition that the temperature of the object is 50°F at 11:07. Since 11:07 is $t = 2$ on our time scale, we can determine k by substituting $T = 50$ and $t = 2$ into (4.2.4) to obtain

$$50 = -20 + 80e^{-2k}$$

(Figure 4.2.2); hence,

$$e^{-2k} = \frac{70}{80} = \frac{7}{8}.$$

Taking logarithms and solving for k yields

$$k = -\frac{1}{2} \ln \frac{7}{8} = \frac{1}{2} \ln \frac{8}{7}.$$

Substituting this into (4.2.4) yields

$$T = -20 + 80e^{-\frac{t}{2} \ln \frac{8}{7}},$$

and the condition $T(\tau) = 72$ implies that

$$72 = -20 + 80e^{-\frac{\tau}{2} \ln \frac{8}{7}},$$

hence,

$$e^{-\frac{\tau}{2} \ln \frac{8}{7}} = \frac{92}{80} = \frac{23}{20}.$$

Taking logarithms and solving for τ yields

$$\tau = -\frac{2 \ln \frac{23}{20}}{\ln \frac{8}{7}} \approx -2.09 \text{ min.}$$

Therefore the object was placed outside about 2 minutes and 5 seconds before 11:05; that is, at 11:02:55.

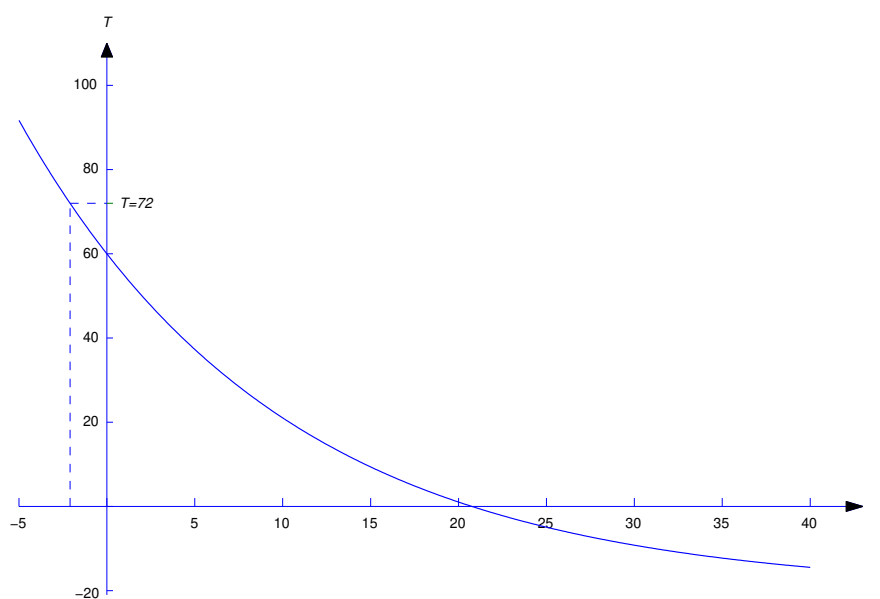


Figure 4.2.2 $T = -20 + 80e^{-\frac{t}{2} \ln \frac{8}{7}}$

Example 4.2.3 A tank initially contains 40 pounds of salt dissolved in 600 gallons of water. Starting at $t_0 = 0$, water that contains $1/2$ pound of salt per gallon is poured into the tank at the rate of 4 gal/min and the mixture is drained from the tank at the same rate (Figure 4.2.3).

- (a) Find a differential equation for the quantity $Q(t)$ of salt in the tank at time $t > 0$, and solve the equation to determine $Q(t)$.
- (b) Find $\lim_{t \rightarrow \infty} Q(t)$.

Solution (a) To find a differential equation for Q , we must use the given information to derive an expression for Q' . But Q' is the rate at which the quantity of salt in the tank changes with respect to time; thus, if *rate in* denotes the rate at which salt enters the tank and *rate out* denotes the rate by which it leaves, then

$$Q' = \text{rate in} - \text{rate out}. \quad (4.2.5)$$

The rate in is

$$\left(\frac{1}{2} \text{ lb/gal}\right) \times (4 \text{ gal/min}) = 2 \text{ lb/min}.$$

Determining the rate out requires a little more thought. We're removing 4 gallons of the mixture per minute, and there are always 600 gallons in the tank; that is, we're removing $1/150$ of the mixture per minute. Since the salt is evenly distributed in the mixture, we are also removing $1/150$ of the salt per minute. Therefore, if there are $Q(t)$ pounds of salt in the tank at time t , the rate out at any time t is $Q(t)/150$. Alternatively, we can arrive at this conclusion by arguing that

$$\begin{aligned} \text{rate out} &= (\text{concentration}) \times (\text{rate of flow out}) \\ &= (\text{lb/gal}) \times (\text{gal/min}) \\ &= \frac{Q(t)}{600} \times 4 = \frac{Q(t)}{150}. \end{aligned}$$

We can now write (4.2.5) as

$$Q' = 2 - \frac{Q}{150}.$$

This first order equation can be rewritten as

$$Q' + \frac{Q}{150} = 2.$$

Since $e^{-t/150}$ is a solution of the complementary equation, the solutions of this equation are of the form $Q = ue^{-t/150}$, where $u'e^{-t/150} = 2$, so $u' = 2e^{t/150}$. Hence,

$$u = 300e^{t/150} + c,$$

so

$$Q = ue^{-t/150} = 300 + ce^{-t/150} \quad (4.2.6)$$

(Figure 4.2.4). Since $Q(0) = 40$, $c = -260$; therefore,

$$Q = 300 - 260e^{-t/150}.$$

Solution (b) From (4.2.6), we see that $\lim_{t \rightarrow \infty} Q(t) = 300$ for any value of $Q(0)$. This is intuitively reasonable, since the incoming solution contains $1/2$ pound of salt per gallon and there are always 600 gallons of water in the tank.

Example 4.2.4 A 500-liter tank initially contains 10 g of salt dissolved in 200 liters of water. Starting at $t_0 = 0$, water that contains $1/4$ g of salt per liter is poured into the tank at the rate of 4 liters/min and the mixture is drained from the tank at the rate of 2 liters/min (Figure 4.2.5). Find a differential equation for the quantity $Q(t)$ of salt in the tank at time t prior to the time when the tank overflows and find the concentration $K(t)$ (g/liter) of salt in the tank at any such time.

Solution We first determine the amount $W(t)$ of solution in the tank at any time t prior to overflow. Since $W(0) = 200$ and we're adding 4 liters/min while removing only 2 liters/min, there's a net gain of 2 liters/min in the tank; therefore,

$$W(t) = 2t + 200.$$

Since $W(150) = 500$ liters (capacity of the tank), this formula is valid for $0 \leq t \leq 150$.

Now let $Q(t)$ be the number of grams of salt in the tank at time t , where $0 \leq t \leq 150$. As in Example 4.2.3,

$$Q' = \text{rate in} - \text{rate out}. \quad (4.2.7)$$

The rate in is

$$\left(\frac{1}{4} \text{ g/liter}\right) \times (4 \text{ liters/min}) = 1 \text{ g/min}. \quad (4.2.8)$$

To determine the rate out, we observe that since the mixture is being removed from the tank at the constant rate of 2 liters/min and there are $2t + 200$ liters in the tank at time t , the fraction of the mixture being removed per minute at time t is

$$\frac{2}{2t + 200} = \frac{1}{t + 100}.$$

We're removing this same fraction of the salt per minute. Therefore, since there are $Q(t)$ grams of salt in the tank at time t ,

$$\text{rate out} = \frac{Q(t)}{t + 100}. \quad (4.2.9)$$

Alternatively, we can arrive at this conclusion by arguing that

$$\begin{aligned} \text{rate out} &= (\text{concentration}) \times (\text{rate of flow out}) = (\text{g/liter}) \times (\text{liters/min}) \\ &= \frac{Q(t)}{2t + 200} \times 2 = \frac{Q(t)}{t + 100}. \end{aligned}$$

Substituting (4.2.8) and (4.2.9) into (4.2.7) yields

$$Q' = 1 - \frac{Q}{t + 100}, \quad \text{so} \quad Q' + \frac{1}{t + 100}Q = 1. \quad (4.2.10)$$

By separation of variables, $1/(t + 100)$ is a solution of the complementary equation, so the solutions of (4.2.10) are of the form

$$Q = \frac{u}{t + 100}, \quad \text{where} \quad \frac{u'}{t + 100} = 1, \quad \text{so} \quad u' = t + 100.$$

Hence,

$$u = \frac{(t + 100)^2}{2} + c. \quad (4.2.11)$$

Since $Q(0) = 10$ and $u = (t + 100)Q$, (4.2.11) implies that

$$(100)(10) = \frac{(100)^2}{2} + c,$$

so

$$c = 100(10) - \frac{(100)^2}{2} = -4000$$

and therefore

$$u = \frac{(t + 100)^2}{2} - 4000.$$

Hence,

$$Q = \frac{u}{t + 100} = \frac{t + 100}{2} - \frac{4000}{t + 100}.$$

Now let $K(t)$ be the concentration of salt at time t . Then

$$K(t) = \frac{1}{4} - \frac{2000}{(t + 100)^2}$$

(Figure 4.2.6).

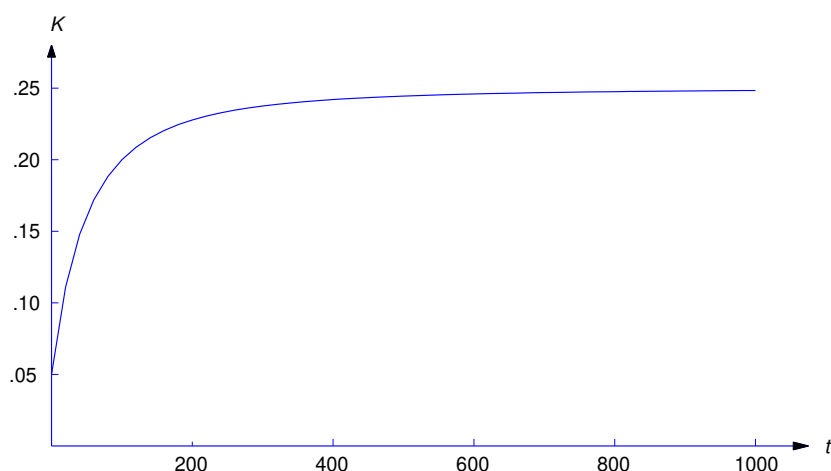


Figure 4.2.6 $K(t) = \frac{1}{4} - \frac{2000}{(t + 100)^2}$

Section 4.3

Example 4.3.1 An object with mass m moves under constant gravitational force through a medium that exerts a resistance with magnitude proportional to the speed of the object. (Recall that the speed of an object is $|v|$, the absolute value of its velocity v .) Find the velocity of the object as a function of t , and find the terminal velocity. Assume that the initial velocity is v_0 .

Solution The total force acting on the object is

$$F = -mg + F_1, \quad (4.3.4)$$

where $-mg$ is the force due to gravity and F_1 is the resisting force of the medium, which has magnitude $k|v|$, where k is a positive constant. If the object is moving downward ($v \leq 0$), the resisting force is upward (Figure 4.3.1(a)), so

$$F_1 = k|v| = k(-v) = -kv.$$

On the other hand, if the object is moving upward ($v \geq 0$), the resisting force is downward (Figure 4.3.1(b)), so

$$F_1 = -k|v| = -kv.$$

Thus, (4.3.4) can be written as

$$F = -mg - kv, \quad (4.3.5)$$

regardless of the sign of the velocity.

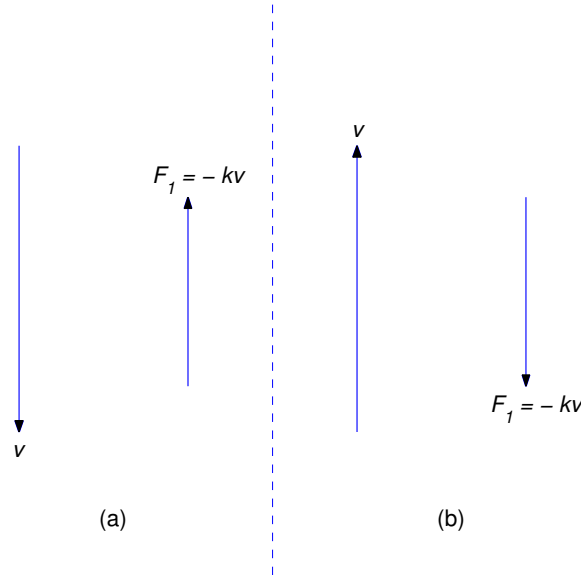


Figure 4.3.1 Resistive forces

From Newton's second law of motion,

$$F = ma = mv',$$

so (4.3.5) yields

$$mv' = -mg - kv,$$

or

$$v' + \frac{k}{m}v = -g. \quad (4.3.6)$$

Since $e^{-kt/m}$ is a solution of the complementary equation, the solutions of (4.3.6) are of the form $v = ue^{-kt/m}$, where $u'e^{-kt/m} = -g$, so $u' = -ge^{kt/m}$. Hence,

$$u = -\frac{mg}{k}e^{kt/m} + c,$$

so

$$v = ue^{-kt/m} = -\frac{mg}{k} + ce^{-kt/m}. \quad (4.3.7)$$

Since $v(0) = v_0$,

$$v_0 = -\frac{mg}{k} + c,$$

so

$$c = v_0 + \frac{mg}{k}$$

and (4.3.7) becomes

$$v = -\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right)e^{-kt/m}.$$

Letting $t \rightarrow \infty$ here shows that the terminal velocity is

$$\lim_{t \rightarrow \infty} v(t) = -\frac{mg}{k},$$

which is independent of the initial velocity v_0 (Figure 4.3.2).

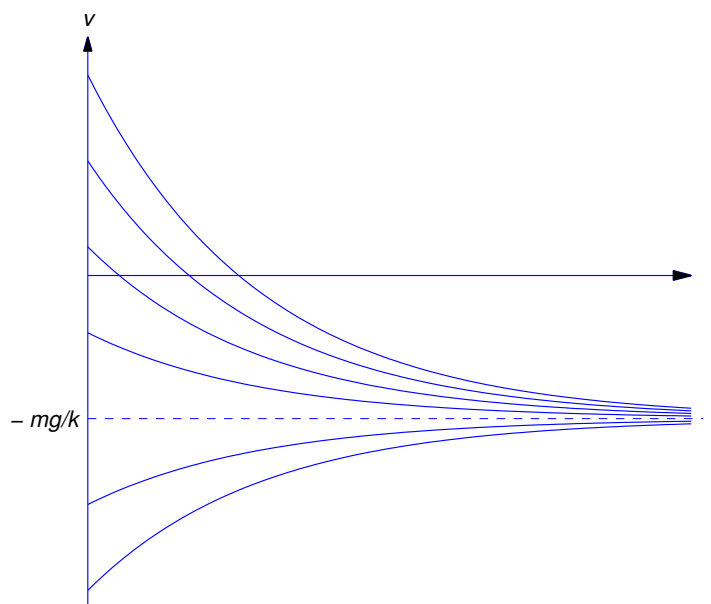


Figure 4.3.2 Solutions of $mv' = -mg - kv$

Example 4.3.2 A 960-lb object is given an initial upward velocity of 60 ft/s near the surface of Earth. The atmosphere resists the motion with a force of 3 lb for each ft/s of speed. Assuming that the only other force acting on the object is constant gravity, find its velocity v as a function of t , and find its terminal velocity.

Solution Since $mg = 960$ and $g = 32$, $m = 960/32 = 30$. The atmospheric resistance is $-3v$ lb if v is expressed in feet per second. Therefore

$$30v' = -960 - 3v,$$

which we rewrite as

$$v' + \frac{1}{10}v = -32.$$

Since $e^{-t/10}$ is a solution of the complementary equation, the solutions of this equation are of the form $v = ue^{-t/10}$, where $u'e^{-t/10} = -32$, so $u' = -32e^{t/10}$. Hence,

$$u = -320e^{t/10} + c,$$

so

$$v = ue^{-t/10} = -320 + ce^{-t/10}. \quad (4.3.8)$$

The initial velocity is 60 ft/s in the upward (positive) direction; hence, $v_0 = 60$. Substituting $t = 0$ and $v = 60$ in (4.3.8) yields

$$60 = -320 + c,$$

so $c = 380$, and (4.3.8) becomes

$$v = -320 + 380e^{-t/10} \text{ ft/s}$$

The terminal velocity is

$$\lim_{t \rightarrow \infty} v(t) = -320 \text{ ft/s.}$$

Example 4.3.3 A 10 kg mass is given an initial velocity $v_0 \leq 0$ near Earth's surface. The only forces acting on it are gravity and atmospheric resistance proportional to the square of the speed. Assuming that the resistance is 8 N if the speed is 2 m/s, find the velocity of the object as a function of t , and find the terminal velocity.

Solution Since the object is falling, the resistance is in the upward (positive) direction. Hence,

$$mv' = -mg + kv^2, \quad (4.3.9)$$

where k is a constant. Since the magnitude of the resistance is 8 N when $v = 2$ m/s,

$$k(2^2) = 8,$$

so $k = 2 \text{ N}\cdot\text{s}^2/\text{m}^2$. Since $m = 10$ and $g = 9.8$, (4.3.9) becomes

$$10v' = -98 + 2v^2 = 2(v^2 - 49). \quad (4.3.10)$$

If $v_0 = -7$, then $v \equiv -7$ for all $t \geq 0$. If $v_0 \neq -7$, we separate variables to obtain

$$\frac{1}{v^2 - 49} v' = \frac{1}{5}, \quad (4.3.11)$$

which is convenient for the required partial fraction expansion

$$\frac{1}{v^2 - 49} = \frac{1}{(v - 7)(v + 7)} = \frac{1}{14} \left[\frac{1}{v - 7} - \frac{1}{v + 7} \right]. \quad (4.3.12)$$

Substituting (4.3.12) into (4.3.11) yields

$$\frac{1}{14} \left[\frac{1}{v - 7} - \frac{1}{v + 7} \right] v' = \frac{1}{5},$$

so

$$\left[\frac{1}{v - 7} - \frac{1}{v + 7} \right] v' = \frac{14}{5}.$$

Integrating this yields

$$\ln |v - 7| - \ln |v + 7| = 14t/5 + C.$$

Therefore

$$\left| \frac{v - 7}{v + 7} \right| = e^C e^{14t/5}.$$

Since Theorem 2.3.1 implies that $(v - 7)/(v + 7)$ can't change sign (why?), we can rewrite the last equation as

$$\frac{v - 7}{v + 7} = c e^{14t/5}, \quad (4.3.13)$$

which is an implicit solution of (4.3.10). Solving this for v yields

$$v = -7 \frac{c + e^{-14t/5}}{c - e^{-14t/5}}. \quad (4.3.14)$$

Since $v(0) = v_0$, (4.3.13) implies that

$$c = \frac{v_0 - 7}{v_0 + 7}.$$

Substituting this into (4.3.14) and simplifying yields

$$v = -7 \frac{v_0(1 + e^{-14t/5}) - 7(1 - e^{-14t/5})}{v_0(1 - e^{-14t/5}) - 7(1 + e^{-14t/5})}.$$

Since $v_0 \leq 0$, v is defined and negative for all $t > 0$. The terminal velocity is

$$\lim_{t \rightarrow \infty} v(t) = -7 \text{ m/s},$$

independent of v_0 . More generally, it can be shown (Exercise 4.3.11) that if v is any solution of (4.3.9) such that $v_0 \leq 0$ then

$$\lim_{t \rightarrow \infty} v(t) = -\sqrt{\frac{mg}{k}}$$

(Figure 4.3.3).

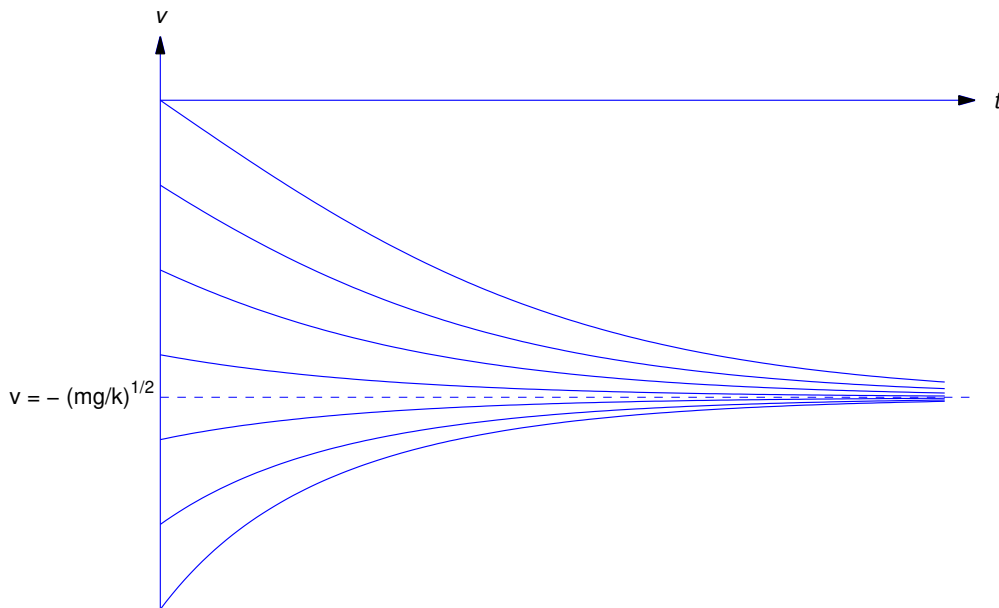


Figure 4.3.3 Solutions of $mv' = -mg + kv^2$, $v(0) = v_0 \leq 0$

Example 4.3.4 A 10-kg mass is launched vertically upward from Earth's surface with an initial velocity of v_0 m/s. The only forces acting on the mass are gravity and atmospheric resistance proportional to the square of the speed. Assuming that the atmospheric resistance is 8 N if the speed is 2 m/s, find the time T required for the mass to reach maximum altitude.

Solution The mass will climb while $v > 0$ and reach its maximum altitude when $v = 0$. Therefore $v > 0$ for $0 \leq t < T$ and $v(T) = 0$. Although the mass of the object and our assumptions concerning the forces acting on it are the same as those in Example 3, (4.3.10) does not apply here, since the resisting force is negative if $v > 0$; therefore, we replace (4.3.10) by

$$10v' = -98 - 2v^2. \quad (4.3.15)$$

Separating variables yields

$$\frac{5}{v^2 + 49} v' = -1,$$

and integrating this yields

$$\frac{5}{7} \tan^{-1} \frac{v}{7} = -t + c.$$

(Recall that $\tan^{-1} u$ is the number θ such that $-\pi/2 < \theta < \pi/2$ and $\tan \theta = u$.) Since $v(0) = v_0$,

$$c = \frac{5}{7} \tan^{-1} \frac{v_0}{7},$$

so v is defined implicitly by

$$\frac{5}{7} \tan^{-1} \frac{v}{7} = -t + \frac{5}{7} \tan^{-1} \frac{v_0}{7}, \quad 0 \leq t \leq T. \quad (4.3.16)$$

Solving this for v yields

$$v = 7 \tan \left(-\frac{7t}{5} + \tan^{-1} \frac{v_0}{7} \right). \quad (4.3.17)$$

Using the identity

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

with $A = \tan^{-1}(v_0/7)$ and $B = 7t/5$, and noting that $\tan(\tan^{-1} \theta) = \theta$, we can simplify (4.3.17) to

$$v = 7 \frac{v_0 - 7 \tan(7t/5)}{7 + v_0 \tan(7t/5)}.$$

Since $v(T) = 0$ and $\tan^{-1}(0) = 0$, (4.3.16) implies that

$$-T + \frac{5}{7} \tan^{-1} \frac{v_0}{7} = 0.$$

Therefore

$$T = \frac{5}{7} \tan^{-1} \frac{v_0}{7}.$$

Since $\tan^{-1}(v_0/7) < \pi/2$ for all v_0 , the time required for the mass to reach its maximum altitude is less than

$$\frac{5\pi}{14} \approx 1.122 \text{ s}$$

regardless of the initial velocity. Figure 4.3.4 shows graphs of v over $[0, T]$ for various values of v_0 .

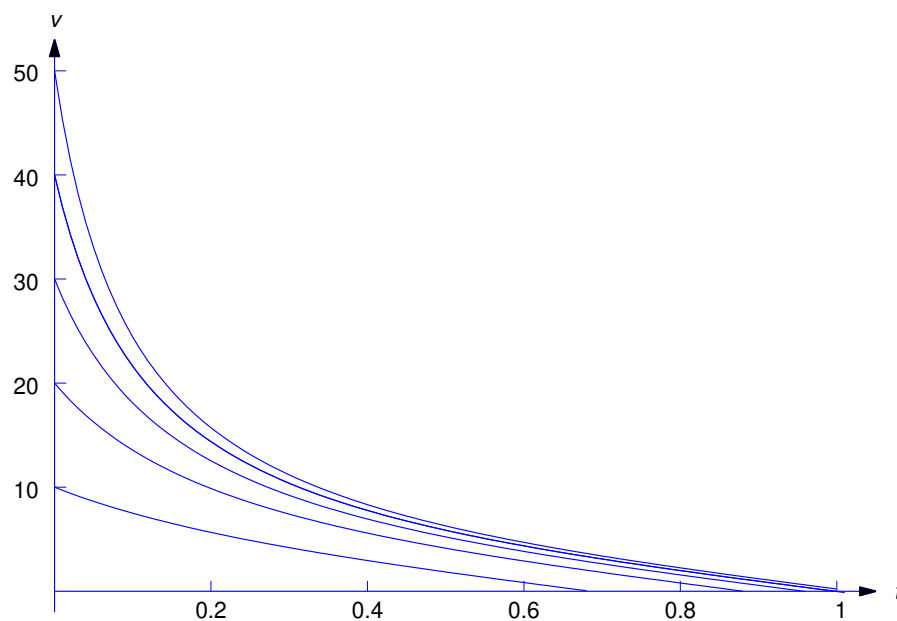


Figure 4.3.4 Solutions of (4.3.15) for various $v_0 > 0$

Section 4.4

Example 4.4.1 [*The Undamped Spring - Mass System*] Consider an object with mass m suspended from a spring and moving vertically. Let y be the displacement of the object from the position it occupies when suspended at rest from the spring (Figure 4.4.2).

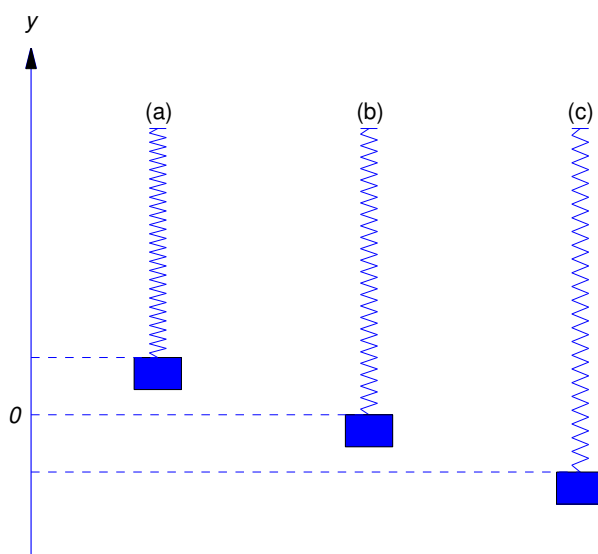


Figure 4.4.2 (a) $y > 0$ (b) $y = 0$ (c) $y < 0$

Assume that if the length of the spring is changed by an amount ΔL (positive or negative), then the spring exerts an opposing force with magnitude $k|\Delta L|$, where k is a positive constant. In Section 6.1 it will be shown that if the mass of the spring is negligible compared to m and no other forces act on the object then Newton's second law of motion implies that

$$my'' = -ky, \quad (4.4.9)$$

which can be written in the form (4.4.7) with $p(y) = ky/m$. This equation can be solved easily by a method that we'll study in Section 5.2, but that method isn't available here. Instead, we'll consider the phase plane equivalent of (4.4.9).

From (4.4.3), we can rewrite (4.4.9) as the separable equation

$$mv \frac{dv}{dy} = -ky.$$

Integrating this yields

$$\frac{mv^2}{2} = -\frac{ky^2}{2} + c,$$

which implies that

$$mv^2 + ky^2 = \rho \quad (4.4.10)$$

($\rho = 2c$). This defines an ellipse in the Poincaré phase plane (Figure 4.4.3).

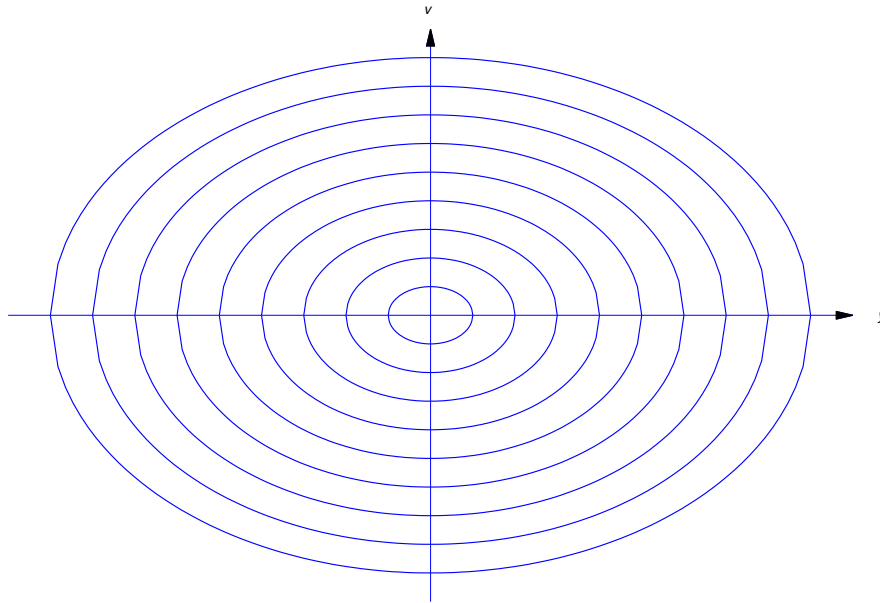


Figure 4.4.3 Trajectories of $my'' + ky = 0$

We can identify ρ by setting $t = 0$ in (4.4.10); thus, $\rho = mv_0^2 + ky_0^2$, where $y_0 = y(0)$ and $v_0 = v(0)$. To determine the maximum and minimum values of y we set $v = 0$ in (4.4.10); thus,

$$y_{\max} = R \quad \text{and} \quad y_{\min} = -R, \quad \text{with } R = \sqrt{\frac{\rho}{k}}. \quad (4.4.11)$$

Equation (4.4.9) has exactly one equilibrium, $\bar{y} = 0$, and it's stable. You can see intuitively why this is so: if the object is displaced in either direction from equilibrium, the spring tries to bring it back.

In this case we can find y explicitly as a function of t . (Don't expect this to happen in more complicated problems!) If $v > 0$ on an interval I , (4.4.10) implies that

$$\frac{dy}{dt} = v = \sqrt{\frac{\rho - ky^2}{m}}$$

on I . This is equivalent to

$$\frac{\sqrt{k}}{\sqrt{\rho - ky^2}} \frac{dy}{dt} = \omega_0, \quad \text{where} \quad \omega_0 = \sqrt{\frac{k}{m}}. \quad (4.4.12)$$

Since

$$\int \frac{\sqrt{k} dy}{\sqrt{\rho - ky^2}} = \sin^{-1} \left(\sqrt{\frac{k}{\rho}} y \right) + c = \sin^{-1} \left(\frac{y}{R} \right) + c$$

(see (4.4.11)), (4.4.12) implies that there's a constant ϕ such that

$$\sin^{-1} \left(\frac{y}{R} \right) = \omega_0 t + \phi$$

or

$$y = R \sin(\omega_0 t + \phi)$$

for all t in I . Although we obtained this function by assuming that $v > 0$, you can easily verify that y satisfies (4.4.9) for all values of t . Thus, the displacement varies periodically between $-R$ and R , with period $T = 2\pi/\omega_0$ (Figure 4.4.4). (If you've taken a course in elementary mechanics you may recognize this as *simple harmonic motion*.)

Example 4.4.2 [*The Undamped Pendulum*] Now we consider the motion of a pendulum with mass m , attached to the end of a weightless rod with length L that rotates on a frictionless axle (Figure 4.4.5). We assume that there's no air resistance.

Let y be the angle measured from the rest position (vertically downward) of the pendulum, as shown in Figure 4.4.5. Newton's second law of motion says that the product of m and the tangential acceleration equals the tangential component of the gravitational force; therefore, from Figure 4.4.5,

$$mLy'' = -mg \sin y,$$

or

$$y'' = -\frac{g}{L} \sin y. \quad (4.4.13)$$

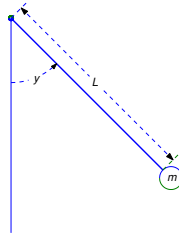


Figure 4.4.5 The undamped pendulum

Since $\sin n\pi = 0$ if n is any integer, (4.4.13) has infinitely many equilibria $\bar{y}_n = n\pi$. If n is even, the mass is directly below the axle (Figure 4.4.6 (a)) and gravity opposes any deviation from the equilibrium. However, if n is odd, the mass is directly above the axle (Figure 4.4.6 (b)) and gravity increases any deviation from the equilibrium. Therefore we conclude on physical grounds that $\bar{y}_{2m} = 2m\pi$ is stable and $\bar{y}_{2m+1} = (2m+1)\pi$ is unstable.

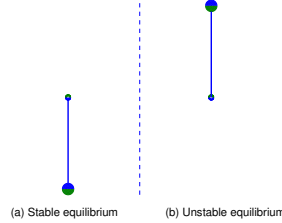


Figure 4.4.6 (a) Stable equilibrium (b) Unstable equilibrium

The phase plane equivalent of (4.4.13) is

$$v \frac{dv}{dy} = -\frac{g}{L} \sin y,$$

where $v = y'$ is the angular velocity of the pendulum. Integrating this yields

$$\frac{v^2}{2} = \frac{g}{L} \cos y + c. \quad (4.4.14)$$

If $v = v_0$ when $y = 0$, then

$$c = \frac{v_0^2}{2} - \frac{g}{L},$$

so (4.4.14) becomes

$$\frac{v^2}{2} = \frac{v_0^2}{2} - \frac{g}{L} (1 - \cos y) = \frac{v_0^2}{2} - \frac{2g}{L} \sin^2 \frac{y}{2},$$

which is equivalent to

$$v^2 = v_0^2 - v_c^2 \sin^2 \frac{y}{2}, \quad (4.4.15)$$

where

$$v_c = 2\sqrt{\frac{g}{L}}.$$

The curves defined by (4.4.15) are the trajectories of (4.4.13). They are periodic with period 2π in y , which isn't surprising, since if $y = y(t)$ is a solution of (4.4.13) then so is $y_n = y(t) + 2n\pi$ for any integer n . Figure 4.4.7 shows trajectories over the interval $[-\pi, \pi]$. From (4.4.15), you can see that if $|v_0| > v_c$ then v is nonzero for all t , which means that the object whirls in the same direction forever, as in Figure 4.4.8. The trajectories associated with this whirling motion are above the upper dashed curve and below the lower dashed curve in Figure 4.4.7. You can also see from (4.4.15) that if $0 < |v_0| < v_c$, then $v = 0$ when $y = \pm y_{\max}$, where

$$y_{\max} = 2 \sin^{-1}(|v_0|/v_c).$$

In this case the pendulum oscillates periodically between $-y_{\max}$ and $y_{\max}$, as shown in Figure 4.4.9. The trajectories associated with this kind of motion are the ovals between the dashed

curves in Figure 4.4.7. It can be shown (see Exercise 4.4.21 for a partial proof) that the period of the oscillation is

$$T = 8 \int_0^{\pi/2} \frac{d\theta}{\sqrt{v_c^2 - v_0^2 \sin^2 \theta}}. \quad (4.4.16)$$

Although this integral can't be evaluated in terms of familiar elementary functions, you can see that it's finite if $|v_0| < v_c$.

The dashed curves in Figure 4.4.7 contain four trajectories. The critical points $(\pi, 0)$ and $(-\pi, 0)$ are the trajectories of the unstable equilibrium solutions $\bar{y} = \pm\pi$. The upper dashed curve connecting (but not including) them is obtained from initial conditions of the form $y(t_0) = 0$, $v(t_0) = v_c$. If y is any solution with this trajectory then

$$\lim_{t \rightarrow \infty} y(t) = \pi \quad \text{and} \quad \lim_{t \rightarrow -\infty} y(t) = -\pi.$$

The lower dashed curve connecting (but not including) them is obtained from initial conditions of the form $y(t_0) = 0$, $v(t_0) = -v_c$. If y is any solution with this trajectory then

$$\lim_{t \rightarrow \infty} y(t) = -\pi \quad \text{and} \quad \lim_{t \rightarrow -\infty} y(t) = \pi.$$

Consistent with this, the integral (4.4.16) diverges to ∞ if $v_0 = \pm v_c$. (Exercise 4.4.21).

Since the dashed curves separate trajectories of whirling solutions from trajectories of oscillating solutions, each of these curves is called a *separatrix*.

In general, if (4.4.7) has both stable and unstable equilibria then the separatrices are the curves given by (4.4.8) that pass through unstable critical points. Thus, if $(\bar{y}, 0)$ is an unstable critical point, then

$$\frac{v^2}{2} + P(y) = P(\bar{y}) \quad (4.4.17)$$

defines a separatrix passing through $(\bar{y}, 0)$.

Example 4.4.3 The equation

$$y'' + y(y - 1) = 0 \quad (4.4.22)$$

is of the form (4.4.18) with $p(y) = y(y - 1)$. Therefore $\bar{y} = 0$ and $\bar{y} = 1$ are the equilibria of (4.4.22). Since

$$\begin{aligned} y(y - 1) &> 0 & \text{if } y < 0 \text{ or } y > 1, \\ &< 0 & \text{if } 0 < y < 1, \end{aligned}$$

$\bar{y} = 0$ is unstable and $\bar{y} = 1$ is stable.

The phase plane equivalent of (4.4.22) is the separable equation

$$v \frac{dv}{dy} + y(y - 1) = 0.$$

Integrating yields

$$\frac{v^2}{2} + \frac{y^3}{3} - \frac{y^2}{2} = C,$$

which we rewrite as

$$v^2 + \frac{1}{3}y^2(2y - 3) = c \quad (4.4.23)$$

after renaming the constant of integration. These are the trajectories of (4.4.22). If y is any solution of (4.4.22), the point $(y(t), v(t))$ moves along the trajectory of y in the direction of increasing y in the upper half plane ($v = y' > 0$), or in the direction of decreasing y in the lower half plane ($v = y' < 0$).

Figure 4.4.10 shows typical trajectories. The dashed curve through the critical point $(0, 0)$, obtained by setting $c = 0$ in (4.4.23), separates the y - v plane into regions that contain different kinds of trajectories; again, we call this curve a *separatrix*. Trajectories in the region bounded by the closed loop **(b)** are closed curves, so solutions associated with them are periodic. Solutions associated with other trajectories are not periodic. If y is any such solution with trajectory not on the separatrix, let (t_-, t_+) be its maximal interval of existence. Both endpoints are finite, and then

$$\begin{aligned} \lim_{t \rightarrow t_+} y(t) &= -\infty, & \lim_{t \rightarrow t_-} y(t) &= -\infty, \\ \lim_{t \rightarrow t_+} v(t) &= -\infty, & \lim_{t \rightarrow t_-} v(t) &= \infty. \end{aligned}$$

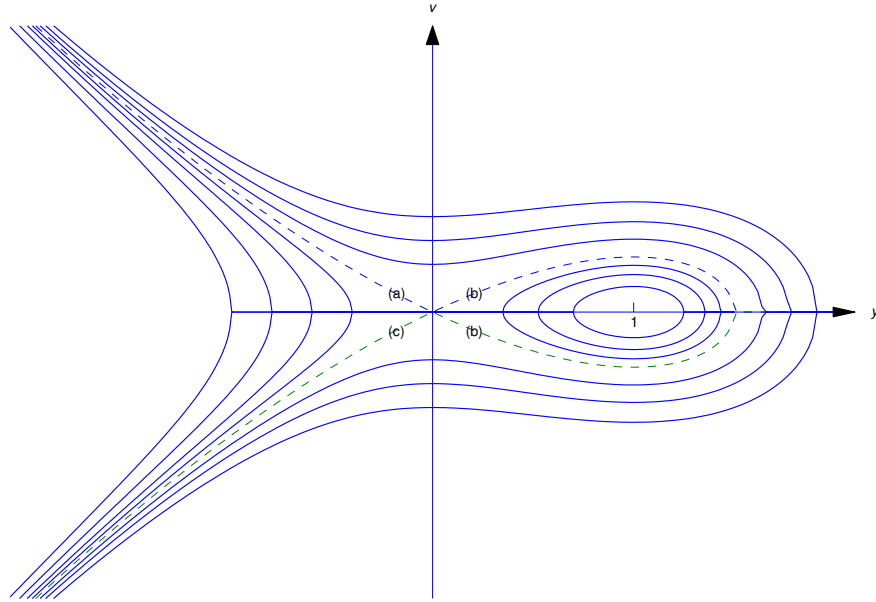


Figure 4.4.10 Trajectories of $y'' + y(y - 1) = 0$

The separatrix contains four trajectories of (4.4.22). One is the point $(0, 0)$, the trajectory of the equilibrium $\bar{y} = 0$. Since distinct trajectories can't intersect, the segments of the separatrix marked **(a)**, **(b)**, and **(c)** – which don't include $(0, 0)$ – are distinct trajectories, none of which can be traversed in finite time. For each solution, let (t_-, t_+) be its maximal interval of existence. On **(a)** and **(c)**, the endpoint at which y tends to $-\infty$ is finite. Solutions with these trajectories have the following asymptotic behavior:

$$\begin{aligned} \lim_{t \rightarrow \infty} y(t) &= 0, & \lim_{t \rightarrow t_-} y(t) &= -\infty, \\ \lim_{t \rightarrow \infty} v(t) &= 0, & \lim_{t \rightarrow t_-} v(t) &= \infty & \text{(on (a))} \\ \lim_{t \rightarrow \infty} y(t) &= 0, & \lim_{t \rightarrow -\infty} y(t) &= 0, \\ \lim_{t \rightarrow \infty} v(t) &= 0, & \lim_{t \rightarrow -\infty} v(t) &= 0 & \text{(on (b))} \\ \lim_{t \rightarrow t_+} y(t) &= -\infty, & \lim_{t \rightarrow -\infty} y(t) &= 0, \\ \lim_{t \rightarrow t_+} v(t) &= -\infty, & \lim_{t \rightarrow -\infty} v(t) &= 0 & \text{(on (c)).} \end{aligned}$$

Example 4.4.4 (The Damped Pendulum) Now we return to the pendulum. If we assume that some mechanism (for example, friction in the axle or atmospheric resistance) opposes the motion of the pendulum with a force proportional to its angular velocity, Newton's second law of motion implies that

$$mLy'' = -cy' - mg \sin y, \quad (4.4.29)$$

where $c > 0$ is the damping constant. (Again, a minor note: the c in (4.4.26) actually corresponds to c/mL in this equation.) To plot a direction field for (4.4.29) we write its phase plane equivalent as

$$\frac{dv}{dy} = -\frac{c}{mL} - \frac{g}{Lv} \sin y.$$

Figure 4.4.15 shows trajectories of four solutions of (4.4.29), all satisfying $y(0) = 0$. For each $m = 0, 1, 2, 3$, imparting the initial velocity $v(0) = v_m$ causes the pendulum to make m complete revolutions and then settle into decaying oscillation about the stable equilibrium $\bar{y} = 2m\pi$.

Section 4.5

Example 4.5.1 For each value of the parameter c , the equation

$$y - cx^2 = 0 \quad (4.5.1)$$

defines a curve in the xy -plane. If $c \neq 0$, the curve is a parabola through the origin, opening upward if $c > 0$ or downward if $c < 0$. If $c = 0$, the curve is the x axis (Figure 4.5.1).

Example 4.5.2 For each value of the parameter c the equation

$$y = x + c \quad (4.5.2)$$

defines a line with slope 1 (Figure 4.5.2).

Example 4.5.3 If $c > 0$, the graph of the equation

$$x^2 + y^2 - c = 0 \quad (4.5.4)$$

is a circle with center at $(0, 0)$ and radius $\sqrt{c}$. If $c = 0$, the graph is the single point $(0, 0)$. (We don't regard a single point as a curve.) If $c < 0$, the equation has no graph. Hence, (4.5.4) defines a one-parameter family of curves for positive values of c . This family consists of all circles centered at $(0, 0)$ (Figure 4.5.3).

Example 4.5.4 The equation

$$x^2 + y^2 + c^2 = 0$$

does not define a one-parameter family of curves, since no (x, y) satisfies the equation if $c \neq 0$, and only the single point $(0, 0)$ satisfies it if $c = 0$. ■

Example 4.5.5 Find a differential equation for the family of curves defined by

$$y = cx^2. \quad (4.5.8)$$

Solution Differentiating (4.5.8) with respect to x yields

$$y' = 2cx.$$

Therefore $c = y'/2x$, and substituting this into (4.5.8) yields

$$y = \frac{xy'}{2}$$

as a differential equation for the family of curves defined by (4.5.8). The graph of any function of the form $y = cx^2$ is an integral curve of this equation. ■

Example 4.5.6

- (a) Try to find a differential equation for the family of lines tangent to the parabola $y = x^2$.
- (b) Find two tangent lines to the parabola $y = x^2$ that pass through $(2, 3)$, and find the points of tangency.

Solution (a) The equation of the line through a given point (x_0, y_0) with slope m is

$$y = y_0 + m(x - x_0). \quad (4.5.9)$$

If (x_0, y_0) is on the parabola, then $y_0 = x_0^2$ and the slope of the tangent line through (x_0, x_0^2) is $m = 2x_0$; hence, (4.5.9) becomes

$$y = x_0^2 + 2x_0(x - x_0),$$

or, equivalently,

$$y = -x_0^2 + 2x_0x. \quad (4.5.10)$$

Here x_0 plays the role of the constant c in Definition 4.5.1; that is, varying x_0 over $(-\infty, \infty)$ produces the family of tangent lines to the parabola $y = x^2$.

Differentiating (4.5.10) with respect to x yields $y' = 2x_0$. We can express x_0 in terms of x and y by rewriting (4.5.10) as

$$x_0^2 - 2x_0x + y = 0$$

and using the quadratic formula to obtain

$$x_0 = x \pm \sqrt{x^2 - y}. \quad (4.5.11)$$

We must choose the plus sign in (4.5.11) if $x < x_0$ and the minus sign if $x > x_0$; thus,

$$x_0 = \left(x + \sqrt{x^2 - y}\right) \quad \text{if } x < x_0$$

and

$$x_0 = \left(x - \sqrt{x^2 - y}\right) \quad \text{if } x > x_0.$$

Since $y' = 2x_0$, this implies that

$$y' = 2\left(x + \sqrt{x^2 - y}\right), \quad \text{if } x < x_0 \quad (4.5.12)$$

and

$$y' = 2\left(x - \sqrt{x^2 - y}\right), \quad \text{if } x > x_0. \quad (4.5.13)$$

Neither (4.5.12) nor (4.5.13) is a differential equation for the family of tangent lines to the parabola $y = x^2$. However, if each tangent line is regarded as consisting of two *tangent half lines* joined at the point of tangency, (4.5.12) is a differential equation for the family of tangent half lines on which x is less than the abscissa of the point of tangency (Figure 4.5.4(a)), while (4.5.13) is a differential equation for the family of tangent half lines on which x is greater than this abscissa (Figure 4.5.4(b)). The parabola $y = x^2$ is also an integral curve of both (4.5.12) and (4.5.13).

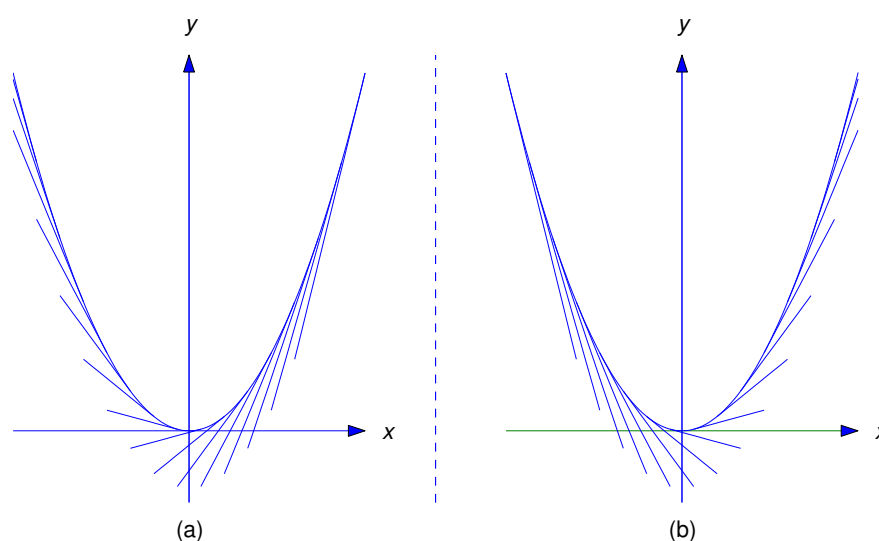


Figure 4.5.4

Solution (b) From (4.5.10) the point $(x, y) = (2, 3)$ is on the tangent line through (x_0, x_0^2) if and only if

$$3 = -x_0^2 + 4x_0,$$

which is equivalent to

$$x_0^2 - 4x_0 + 3 = (x_0 - 3)(x_0 - 1) = 0.$$

Letting $x_0 = 3$ in (4.5.10) shows that $(2, 3)$ is on the line

$$y = -9 + 6x,$$

which is tangent to the parabola at $(x_0, x_0^2) = (3, 9)$, as shown in Figure 4.5.5

Letting $x_0 = 1$ in (4.5.10) shows that $(2, 3)$ is on the line

$$y = -1 + 2x,$$

which is tangent to the parabola at $(x_0, x_0^2) = (1, 1)$, as shown in Figure 4.5.5.

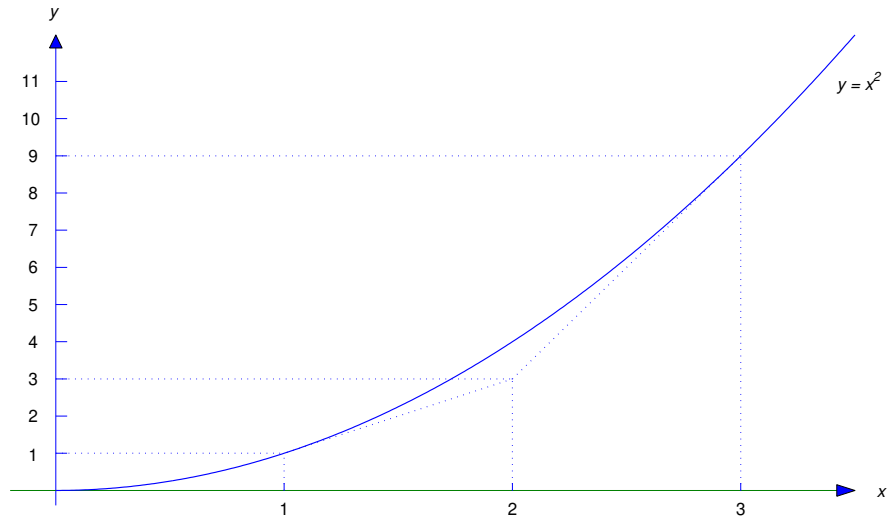


Figure 4.5.5

Example 4.5.7 Find curves $y = y(x)$ such that every point $(x_0, y(x_0))$ on the curve is the midpoint of the line segment with endpoints on the coordinate axes and tangent to the curve at $(x_0, y(x_0))$ (Figure 4.5.6).

Solution The equation of the line tangent to the curve at $P = (x_0, y(x_0))$ is

$$y = y(x_0) + y'(x_0)(x - x_0).$$

If we denote the x and y intercepts of the tangent line by x_I and y_I (Figure 4.5.6), then

$$0 = y(x_0) + y'(x_0)(x_I - x_0) \quad (4.5.14)$$

and

$$y_I = y(x_0) - y'(x_0)x_0. \quad (4.5.15)$$

From Figure 4.5.6, P is the midpoint of the line segment connecting $(x_I, 0)$ and $(0, y_I)$ if and only if $x_I = 2x_0$ and $y_I = 2y(x_0)$. Substituting the first of these conditions into (4.5.14) or the second into (4.5.15) yields

$$y(x_0) + y'(x_0)x_0 = 0.$$

Since x_0 is arbitrary we drop the subscript and conclude that $y = y(x)$ satisfies

$$y + xy' = 0,$$

which can be rewritten as

$$(xy)' = 0.$$

Integrating yields $xy = c$, or

$$y = \frac{c}{x}.$$

If $c = 0$ this curve is the line $y = 0$, which does not satisfy the geometric requirements imposed by the problem; thus, $c \neq 0$, and the solutions define a family of hyperbolas (Figure 4.5.7).

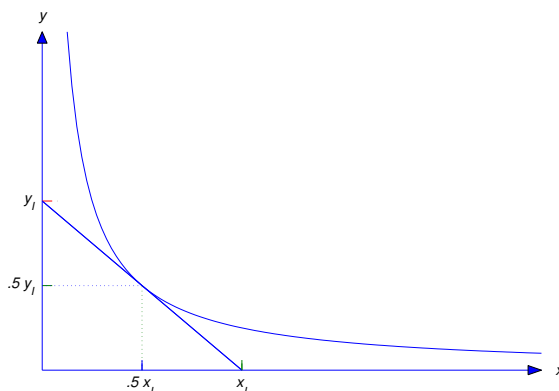


Figure 4.5.6

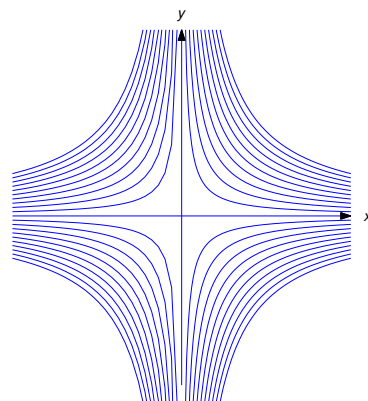


Figure 4.5.7

Example 4.5.8 Find curves $y = y(x)$ such that the tangent line to the curve at any point $(x_0, y(x_0))$ intersects the x -axis at $(x_0^2, 0)$. Figure 4.5.8 illustrates the situation in the case where the curve is in the first quadrant and $0 < x < 1$.

Solution The equation of the line tangent to the curve at $(x_0, y(x_0))$ is

$$y = y(x_0) + y'(x_0)(x - x_0).$$

Since $(x_0^2, 0)$ is on the tangent line,

$$0 = y(x_0) + y'(x_0)(x_0^2 - x_0).$$

Since x_0 is arbitrary we drop the subscript and conclude that $y = y(x)$ satisfies

$$y + y'(x^2 - x) = 0.$$

Therefore

$$\frac{y'}{y} = -\frac{1}{x^2 - x} = -\frac{1}{x(x - 1)} = \frac{1}{x} - \frac{1}{x - 1},$$

so

$$\ln |y| = \ln |x| - \ln |x - 1| + k = \ln \left| \frac{x}{x - 1} \right| + k,$$

and

$$y = \frac{cx}{x - 1}.$$

If $c = 0$, the graph of this function is the x -axis. If $c \neq 0$, it's a hyperbola with vertical asymptote $x = 1$ and horizontal asymptote $y = c$. Figure 4.5.9 shows the graphs for $c \neq 0$.

Example 4.5.9 Find the orthogonal trajectories of the family of circles

$$x^2 + y^2 = c^2 \quad (c > 0). \quad (4.5.17)$$

Solution To find a differential equation for the family of circles we differentiate (4.5.17) implicitly with respect to x to obtain

$$2x + 2yy' = 0,$$

or

$$y' = -\frac{x}{y}.$$

Therefore the integral curves of

$$y' = \frac{y}{x}$$

are orthogonal trajectories of the given family. We leave it to you to verify that the general solution of this equation is

$$y = kx,$$

where k is an arbitrary constant. This is the equation of a nonvertical line through $(0, 0)$. The y axis is also an orthogonal trajectory of the given family. Therefore every line through the origin is an orthogonal trajectory of the given family (4.5.17) (Figure 4.5.11). This is consistent with the theorem of plane geometry which states that a diameter of a circle and a tangent line to the circle at the end of the diameter are perpendicular.

Example 4.5.10 Find the orthogonal trajectories of the family of hyperbolas

$$xy = c \quad (c \neq 0) \tag{4.5.18}$$

(Figure 4.5.7).

Solution Differentiating (4.5.18) implicitly with respect to x yields

$$y + xy' = 0,$$

or

$$y' = -\frac{y}{x};$$

thus, the integral curves of

$$y' = \frac{x}{y}$$

are orthogonal trajectories of the given family. Separating variables yields

$$y'y = x$$

and integrating yields

$$y^2 - x^2 = k,$$

which is the equation of a hyperbola if $k \neq 0$, or of the lines $y = x$ and $y = -x$ if $k = 0$ (Figure 4.5.12).

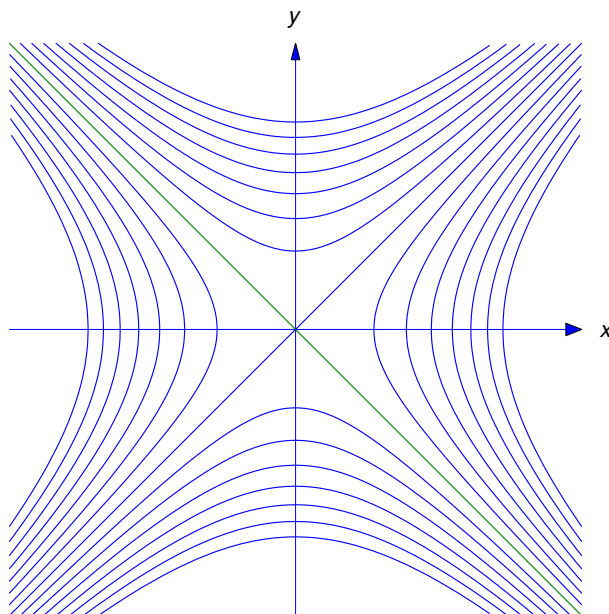


Figure 4.5.12 Orthogonal trajectories of the hyperbolas $xy = c$

Example 4.5.11 Find the orthogonal trajectories of the family of circles defined by

$$(x - c)^2 + y^2 = c^2 \quad (c \neq 0). \quad (4.5.19)$$

These circles are centered on the x -axis and tangent to the y -axis (Figure 4.5.13(a)).

Solution Multiplying out the left side of (4.5.19) yields

$$x^2 - 2cx + y^2 = 0, \quad (4.5.20)$$

and differentiating this implicitly with respect to x yields

$$2(x - c) + 2yy' = 0. \quad (4.5.21)$$

From (4.5.20),

$$c = \frac{x^2 + y^2}{2x},$$

so

$$x - c = x - \frac{x^2 + y^2}{2x} = \frac{x^2 - y^2}{2x}.$$

Substituting this into (4.5.21) and solving for y' yields

$$y' = \frac{y^2 - x^2}{2xy}. \quad (4.5.22)$$

The curves defined by (4.5.19) are integral curves of (4.5.22), and the integral curves of

$$y' = \frac{2xy}{x^2 - y^2}$$

are orthogonal trajectories of the family (4.5.19). This is a homogeneous nonlinear equation, which we studied in Section 2.4. Substituting $y = ux$ yields

$$u'x + u = \frac{2x(ux)}{x^2 - (ux)^2} = \frac{2u}{1 - u^2},$$

so

$$u'x = \frac{2u}{1 - u^2} - u = \frac{u(u^2 + 1)}{1 - u^2},$$

Separating variables yields

$$\frac{1 - u^2}{u(u^2 + 1)} u' = \frac{1}{x},$$

or, equivalently,

$$\left[\frac{1}{u} - \frac{2u}{u^2 + 1} \right] u' = \frac{1}{x}.$$

Therefore

$$\ln |u| - \ln(u^2 + 1) = \ln |x| + k.$$

By substituting $u = y/x$, we see that

$$\ln |y| - \ln |x| - \ln(x^2 + y^2) + \ln(x^2) = \ln |x| + k,$$

which, since $\ln(x^2) = 2 \ln |x|$, is equivalent to

$$\ln |y| - \ln(x^2 + y^2) = k,$$

or

$$|y| = e^k (x^2 + y^2).$$

To see what these curves are we rewrite this equation as

$$x^2 + |y|^2 - e^{-k} |y| = 0$$

and complete the square to obtain

$$x^2 + (|y| - e^{-k}/2)^2 = (e^{-k}/2)^2.$$

This can be rewritten as

$$x^2 + (y - h)^2 = h^2,$$

where

$$h = \begin{cases} \frac{e^{-k}}{2} & \text{if } y \geq 0, \\ -\frac{e^{-k}}{2} & \text{if } y \leq 0. \end{cases}$$

Thus, the orthogonal trajectories are circles centered on the y axis and tangent to the x axis (Figure 4.5.13(b)). The circles for which $h > 0$ are above the x -axis, while those for which $h < 0$ are below.

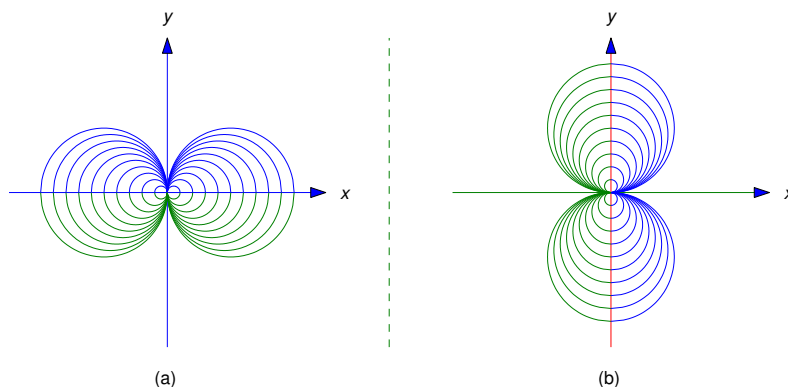


Figure 5.13 (a) The circles $(x - c)^2 + y^2 = c^2$ (b) The circles $x^2 + (y - h)^2 = h^2$

Section 5.1

Example 5.1.1 The coefficients of y' and y in

$$y'' - y = 0 \quad (5.1.4)$$

are the constant functions $p \equiv 0$ and $q \equiv -1$, which are continuous on $(-\infty, \infty)$. Therefore Theorem 5.1.1 implies that every initial value problem for (5.1.4) has a unique solution on $(-\infty, \infty)$.

- (a) Verify that $y_1 = e^x$ and $y_2 = e^{-x}$ are solutions of (5.1.4) on $(-\infty, \infty)$.
- (b) Verify that if c_1 and c_2 are arbitrary constants, $y = c_1 e^x + c_2 e^{-x}$ is a solution of (5.1.4) on $(-\infty, \infty)$.
- (c) Solve the initial value problem

$$y'' - y = 0, \quad y(0) = 1, \quad y'(0) = 3. \quad (5.1.5)$$

Solution (a) If $y_1 = e^x$ then $y'_1 = e^x$ and $y''_1 = e^x = y_1$, so $y''_1 - y_1 = 0$. If $y_2 = e^{-x}$, then $y'_2 = -e^{-x}$ and $y''_2 = e^{-x} = y_2$, so $y''_2 - y_2 = 0$.

Solution (b) If

$$y = c_1 e^x + c_2 e^{-x} \quad (5.1.6)$$

then

$$y' = c_1 e^x - c_2 e^{-x} \quad (5.1.7)$$

and

$$y'' = c_1 e^x + c_2 e^{-x},$$

so

$$\begin{aligned} y'' - y &= (c_1 e^x + c_2 e^{-x}) - (c_1 e^x + c_2 e^{-x}) \\ &= c_1(e^x - e^x) + c_2(e^{-x} - e^{-x}) = 0 \end{aligned}$$

for all x . Therefore $y = c_1 e^x + c_2 e^{-x}$ is a solution of (5.1.4) on $(-\infty, \infty)$.

Solution (c) We can solve (5.1.5) by choosing c_1 and c_2 in (5.1.6) so that $y(0) = 1$ and $y'(0) = 3$. Setting $x = 0$ in (5.1.6) and (5.1.7) shows that this is equivalent to

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_1 - c_2 &= 3. \end{aligned}$$

Solving these equations yields $c_1 = 2$ and $c_2 = -1$. Therefore $y = 2e^x - e^{-x}$ is the unique solution of (5.1.5) on $(-\infty, \infty)$.

Example 5.1.2 Let ω be a positive constant. The coefficients of y' and y in

$$y'' + \omega^2 y = 0 \tag{5.1.8}$$

are the constant functions $p \equiv 0$ and $q \equiv \omega^2$, which are continuous on $(-\infty, \infty)$. Therefore Theorem 5.1.1 implies that every initial value problem for (5.1.8) has a unique solution on $(-\infty, \infty)$.

- (a) Verify that $y_1 = \cos \omega x$ and $y_2 = \sin \omega x$ are solutions of (5.1.8) on $(-\infty, \infty)$.
- (b) Verify that if c_1 and c_2 are arbitrary constants then $y = c_1 \cos \omega x + c_2 \sin \omega x$ is a solution of (5.1.8) on $(-\infty, \infty)$.
- (c) Solve the initial value problem

$$y'' + \omega^2 y = 0, \quad y(0) = 1, \quad y'(0) = 3. \tag{5.1.9}$$

Solution (a) If $y_1 = \cos \omega x$ then $y'_1 = -\omega \sin \omega x$ and $y''_1 = -\omega^2 \cos \omega x = -\omega^2 y_1$, so $y''_1 + \omega^2 y_1 = 0$. If $y_2 = \sin \omega x$ then, $y'_2 = \omega \cos \omega x$ and $y''_2 = -\omega^2 \sin \omega x = -\omega^2 y_2$, so $y''_2 + \omega^2 y_2 = 0$.

Solution (b) If

$$y = c_1 \cos \omega x + c_2 \sin \omega x \tag{5.1.10}$$

then

$$y' = \omega(-c_1 \sin \omega x + c_2 \cos \omega x) \tag{5.1.11}$$

and

$$y'' = -\omega^2(c_1 \cos \omega x + c_2 \sin \omega x),$$

so

$$\begin{aligned} y'' + \omega^2 y &= -\omega^2(c_1 \cos \omega x + c_2 \sin \omega x) + \omega^2(c_1 \cos \omega x + c_2 \sin \omega x) \\ &= c_1 \omega^2(-\cos \omega x + \cos \omega x) + c_2 \omega^2(-\sin \omega x + \sin \omega x) = 0 \end{aligned}$$

for all x . Therefore $y = c_1 \cos \omega x + c_2 \sin \omega x$ is a solution of (5.1.8) on $(-\infty, \infty)$.

Solution (c) To solve (5.1.9), we must choose c_1 and c_2 in (5.1.10) so that $y(0) = 1$ and $y'(0) = 3$. Setting $x = 0$ in (5.1.10) and (5.1.11) shows that $c_1 = 1$ and $c_2 = 3/\omega$. Therefore

$$y = \cos \omega x + \frac{3}{\omega} \sin \omega x$$

is the unique solution of (5.1.9) on $(-\infty, \infty)$. ■

Example 5.1.3 The equation

$$x^2 y'' + xy' - 4y = 0 \quad (5.1.13)$$

has the form of the differential equation in (5.1.12), with $P_0(x) = x^2$, $P_1(x) = x$, and $P_2(x) = -4$, which are all continuous on $(-\infty, \infty)$. However, since $P_0(0) = 0$ we must consider solutions of (5.1.13) on $(-\infty, 0)$ and $(0, \infty)$. Since P_0 has no zeros on these intervals, Theorem 5.1.1 implies that the initial value problem

$$x^2 y'' + xy' - 4y = 0, \quad y(x_0) = k_0, \quad y'(x_0) = k_1$$

has a unique solution on $(0, \infty)$ if $x_0 > 0$, or on $(-\infty, 0)$ if $x_0 < 0$.

- (a) Verify that $y_1 = x^2$ is a solution of (5.1.13) on $(-\infty, \infty)$ and $y_2 = 1/x^2$ is a solution of (5.1.13) on $(-\infty, 0)$ and $(0, \infty)$.
- (b) Verify that if c_1 and c_2 are any constants then $y = c_1 x^2 + c_2/x^2$ is a solution of (5.1.13) on $(-\infty, 0)$ and $(0, \infty)$.
- (c) Solve the initial value problem

$$x^2 y'' + xy' - 4y = 0, \quad y(1) = 2, \quad y'(1) = 0. \quad (5.1.14)$$

- (d) Solve the initial value problem

$$x^2 y'' + xy' - 4y = 0, \quad y(-1) = 2, \quad y'(-1) = 0. \quad (5.1.15)$$

Solution (a) If $y_1 = x^2$ then $y'_1 = 2x$ and $y''_1 = 2$, so

$$x^2 y''_1 + xy'_1 - 4y_1 = x^2(2) + x(2x) - 4x^2 = 0$$

for x in $(-\infty, \infty)$. If $y_2 = 1/x^2$, then $y'_2 = -2/x^3$ and $y''_2 = 6/x^4$, so

$$x^2 y''_2 + xy'_2 - 4y_2 = x^2 \left(\frac{6}{x^4} \right) - x \left(\frac{2}{x^3} \right) - \frac{4}{x^2} = 0$$

for x in $(-\infty, 0)$ or $(0, \infty)$.

Solution (b) If

$$y = c_1 x^2 + \frac{c_2}{x^2} \quad (5.1.16)$$

then

$$y' = 2c_1 x - \frac{2c_2}{x^3} \quad (5.1.17)$$

and

$$y'' = 2c_1 + \frac{6c_2}{x^4},$$

so

$$\begin{aligned} x^2 y'' + xy' - 4y &= x^2 \left(2c_1 + \frac{6c_2}{x^4} \right) + x \left(2c_1 x - \frac{2c_2}{x^3} \right) - 4 \left(c_1 x^2 + \frac{c_2}{x^2} \right) \\ &= c_1 (2x^2 + 2x^2 - 4x^2) + c_2 \left(\frac{6}{x^2} - \frac{2}{x^2} - \frac{4}{x^2} \right) \\ &= c_1 \cdot 0 + c_2 \cdot 0 = 0 \end{aligned}$$

for x in $(-\infty, 0)$ or $(0, \infty)$.

Solution (c) To solve (5.1.14), we choose c_1 and c_2 in (5.1.16) so that $y(1) = 2$ and $y'(1) = 0$. Setting $x = 1$ in (5.1.16) and (5.1.17) shows that this is equivalent to

$$\begin{aligned} c_1 + c_2 &= 2 \\ 2c_1 - 2c_2 &= 0. \end{aligned}$$

Solving these equations yields $c_1 = 1$ and $c_2 = 1$. Therefore $y = x^2 + 1/x^2$ is the unique solution of (5.1.14) on $(0, \infty)$.

Solution (d) We can solve (5.1.15) by choosing c_1 and c_2 in (5.1.16) so that $y(-1) = 2$ and $y'(-1) = 0$. Setting $x = -1$ in (5.1.16) and (5.1.17) shows that this is equivalent to

$$\begin{aligned} c_1 + c_2 &= 2 \\ -2c_1 + 2c_2 &= 0. \end{aligned}$$

Solving these equations yields $c_1 = 1$ and $c_2 = 1$. Therefore $y = x^2 + 1/x^2$ is the unique solution of (5.1.15) on $(-\infty, 0)$. ■

Although the *formulas* for the solutions of (5.1.14) and (5.1.15) are both $y = x^2 + 1/x^2$, you should not conclude that these two initial value problems have the same solution. Remember that a solution of an initial value problem is defined *on an interval that contains the initial point*; therefore, the solution of (5.1.14) is $y = x^2 + 1/x^2$ *on the interval* $(0, \infty)$, which contains the initial point $x_0 = 1$, while the solution of (5.1.15) is $y = x^2 + 1/x^2$ *on the interval* $(-\infty, 0)$, which contains the initial point $x_0 = -1$.

Example 5.1.4

- (a) Since $e^x/e^{-x} = e^{2x}$ is nonconstant, Theorem 5.1.3 implies that $y = c_1 e^x + c_2 e^{-x}$ is the general solution of $y'' - y = 0$ on $(-\infty, \infty)$.
- (b) Since $\cos \omega x / \sin \omega x = \cot \omega x$ is nonconstant, Theorem 5.1.3 implies that $y = c_1 \cos \omega x + c_2 \sin \omega x$ is the general solution of $y'' + \omega^2 y = 0$ on $(-\infty, \infty)$.
- (c) Since $x^2/x^{-2} = x^4$ is nonconstant, Theorem 5.1.3 implies that $y = c_1 x^2 + c_2/x^2$ is the general solution of $x^2 y'' + x y' - 4y = 0$ on $(-\infty, 0)$ and $(0, \infty)$.

Example 5.1.5 Verify Abel's formula for the following differential equations and the corresponding solutions, from Examples 5.1.1, 5.1.2, and 5.1.3:

- (a) $y'' - y = 0$; $y_1 = e^x$, $y_2 = e^{-x}$
- (b) $y'' + \omega^2 y = 0$; $y_1 = \cos \omega x$, $y_2 = \sin \omega x$
- (c) $x^2 y'' + x y' - 4y = 0$; $y_1 = x^2$, $y_2 = 1/x^2$

Solution (a) Since $p \equiv 0$, we can verify Abel's formula by showing that W is constant, which is true, since

$$W(x) = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = e^x(-e^{-x}) - e^x e^{-x} = -2$$

for all x .

Solution (b) Again, since $p \equiv 0$, we can verify Abel's formula by showing that W is constant, which is true, since

$$\begin{aligned} W(x) &= \begin{vmatrix} \cos \omega x & \sin \omega x \\ -\omega \sin \omega x & \omega \cos \omega x \end{vmatrix} \\ &= \cos \omega x (\omega \cos \omega x) - (-\omega \sin \omega x) \sin \omega x \\ &= \omega (\cos^2 \omega x + \sin^2 \omega x) = \omega \end{aligned}$$

for all x .

Solution (c) Computing the Wronskian of $y_1 = x^2$ and $y_2 = 1/x^2$ directly yields

$$W = \begin{vmatrix} x^2 & 1/x^2 \\ 2x & -2/x^3 \end{vmatrix} = x^2 \left(-\frac{2}{x^3} \right) - 2x \left(\frac{1}{x^2} \right) = -\frac{4}{x}. \quad (5.1.31)$$

To verify Abel's formula we rewrite the differential equation as

$$y'' + \frac{1}{x}y' - \frac{4}{x^2}y = 0$$

to see that $p(x) = 1/x$. If x_0 and x are either both in $(-\infty, 0)$ or both in $(0, \infty)$ then

$$\int_{x_0}^x p(t) dt = \int_{x_0}^x \frac{dt}{t} = \ln \left(\frac{x}{x_0} \right),$$

so Abel's formula becomes

$$\begin{aligned} W(x) &= W(x_0)e^{-\ln(x/x_0)} = W(x_0)\frac{x_0}{x} \\ &= -\left(\frac{4}{x_0}\right)\left(\frac{x_0}{x}\right) \quad \text{from (5.1.31)} \\ &= -\frac{4}{x}, \end{aligned}$$

which is consistent with (5.1.31). ■

Section 5.2

Example 5.2.1

(a) Find the general solution of

$$y'' + 6y' + 5y = 0. \quad (5.2.4)$$

(b) Solve the initial value problem

$$y'' + 6y' + 5y = 0, \quad y(0) = 3, \quad y'(0) = -1. \quad (5.2.5)$$

Solution (a) The characteristic polynomial of (5.2.4) is

$$p(r) = r^2 + 6r + 5 = (r + 1)(r + 5).$$

Since $p(-1) = p(-5) = 0$, $y_1 = e^{-x}$ and $y_2 = e^{-5x}$ are solutions of (5.2.4). Since $y_2/y_1 = e^{-4x}$ is nonconstant, 5.1.6 implies that the general solution of (5.2.4) is

$$y = c_1 e^{-x} + c_2 e^{-5x}. \quad (5.2.6)$$

Solution (b) We must determine c_1 and c_2 in (5.2.6) so that y satisfies the initial conditions in (5.2.5). Differentiating (5.2.6) yields

$$y' = -c_1 e^{-x} - 5c_2 e^{-5x}. \quad (5.2.7)$$

Imposing the initial conditions $y(0) = 3$, $y'(0) = -1$ in (5.2.6) and (5.2.7) yields

$$\begin{aligned} c_1 + c_2 &= 3 \\ -c_1 - 5c_2 &= -1. \end{aligned}$$

The solution of this system is $c_1 = 7/2$, $c_2 = -1/2$. Therefore the solution of (5.2.5) is

$$y = \frac{7}{2}e^{-x} - \frac{1}{2}e^{-5x}.$$

Figure 5.2.1 is a graph of this solution.

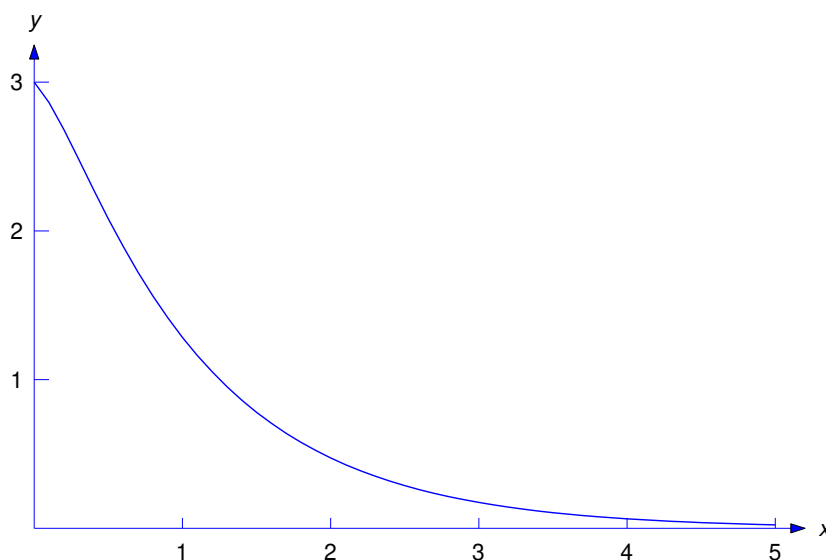


Figure 5.2.1 $y = \frac{7}{2}e^{-x} - \frac{1}{2}e^{-5x}$

Example 5.2.2

(a) Find the general solution of

$$y'' + 6y' + 9y = 0. \quad (5.2.8)$$

(b) Solve the initial value problem

$$y'' + 6y' + 9y = 0, \quad y(0) = 3, \quad y'(0) = -1. \quad (5.2.9)$$

Solution (a) The characteristic polynomial of (5.2.8) is

$$p(r) = r^2 + 6r + 9 = (r + 3)^2,$$

so the characteristic equation has the repeated real root $r_1 = -3$. Therefore $y_1 = e^{-3x}$ is a solution of (5.2.8). Since the characteristic equation has no other roots, (5.2.8) has no other solutions of the form e^{rx} . We look for solutions of the form $y = uy_1 = ue^{-3x}$, where u is a function that we'll now determine. (This should remind you of the method of variation of parameters used in Section 2.1 to solve the nonhomogeneous equation $y' + p(x)y = f(x)$, given a solution y_1 of the complementary equation $y' + p(x)y = 0$. It's also a special case of a method called *reduction of order* that we'll study in Section 5.6. For other ways to obtain a second solution of (5.2.8) that's not a multiple of e^{-3x} , see Exercises 5.1.5.1.9, 5.1.5.1.12, and 5.2.33.

If $y = ue^{-3x}$, then

$$y' = u'e^{-3x} - 3ue^{-3x} \quad \text{and} \quad y'' = u''e^{-3x} - 6u'e^{-3x} + 9ue^{-3x},$$

so

$$\begin{aligned} y'' + 6y' + 9y &= e^{-3x} [(u'' - 6u' + 9u) + 6(u' - 3u) + 9u] \\ &= e^{-3x} [u'' - (6 - 6)u' + (9 - 18 + 9)u] = u''e^{-3x}. \end{aligned}$$

Therefore $y = ue^{-3x}$ is a solution of (5.2.8) if and only if $u'' = 0$, which is equivalent to $u = c_1 + c_2x$, where c_1 and c_2 are constants. Therefore any function of the form

$$y = e^{-3x}(c_1 + c_2x) \quad (5.2.10)$$

is a solution of (5.2.8). Letting $c_1 = 1$ and $c_2 = 0$ yields the solution $y_1 = e^{-3x}$ that we already knew. Letting $c_1 = 0$ and $c_2 = 1$ yields the second solution $y_2 = xe^{-3x}$. Since $y_2/y_1 = x$ is nonconstant, 5.1.6 implies that $\{y_1, y_2\}$ is fundamental set of solutions of (5.2.8), and (5.2.10) is the general solution.

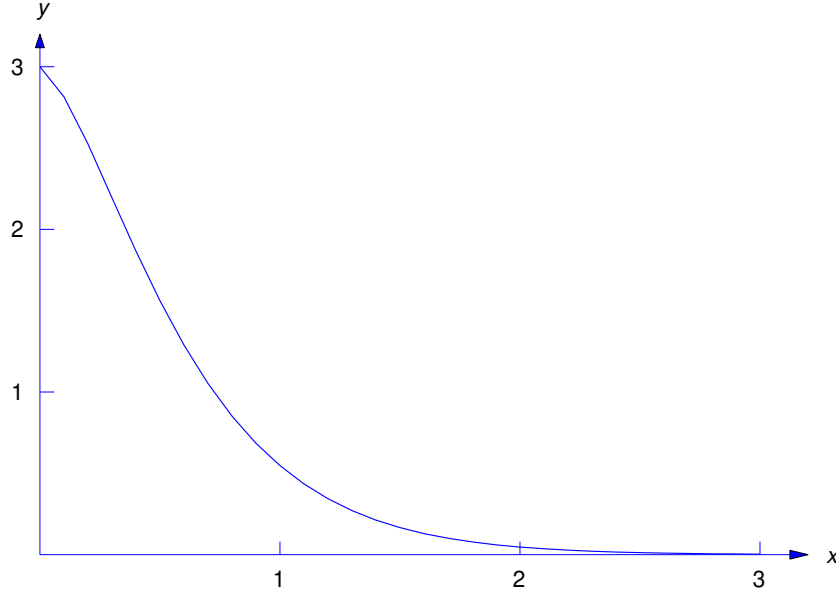
Solution (b) Differentiating (5.2.10) yields

$$y' = -3e^{-3x}(c_1 + c_2x) + c_2e^{-3x}. \quad (5.2.11)$$

Imposing the initial conditions $y(0) = 3$, $y'(0) = -1$ in (5.2.10) and (5.2.11) yields $c_1 = 3$ and $-3c_1 + c_2 = -1$, so $c_2 = 8$. Therefore the solution of (5.2.9) is

$$y = e^{-3x}(3 + 8x).$$

Figure 5.2.2 is a graph of this solution.

Figure 5.2.2 $y = e^{-3x}(3 + 8x)$ **Example 5.2.3**

(a) Find the general solution of

$$y'' + 4y' + 13y = 0. \quad (5.2.14)$$

(b) Solve the initial value problem

$$y'' + 4y' + 13y = 0, \quad y(0) = 2, \quad y'(0) = -3. \quad (5.2.15)$$

Solution (a) The characteristic polynomial of (5.2.14) is

$$p(r) = r^2 + 4r + 13 = r^2 + 4r + 4 + 9 = (r + 2)^2 + 9.$$

The roots of the characteristic equation are $r_1 = -2 + 3i$ and $r_2 = -2 - 3i$. By analogy with Case 1, it's reasonable to expect that $e^{(-2+3i)x}$ and $e^{(-2-3i)x}$ are solutions of (5.2.14). This is true (see Exercise 5.2.34); however, there are difficulties here, since you are probably not familiar with exponential functions with complex arguments, and even if you are, it's inconvenient to work with them, since they are complex-valued. We'll take a simpler approach, which we motivate as follows: the exponential notation suggests that

$$e^{(-2+3i)x} = e^{-2x}e^{3ix} \quad \text{and} \quad e^{(-2-3i)x} = e^{-2x}e^{-3ix},$$

so even though we haven't defined e^{3ix} and e^{-3ix} , it's reasonable to expect that every linear combination of $e^{(-2+3i)x}$ and $e^{(-2-3i)x}$ can be written as $y = ue^{-2x}$, where u depends upon x . To determine u , we note that if $y = ue^{-2x}$ then

$$y' = u'e^{-2x} - 2ue^{-2x} \quad \text{and} \quad y'' = u''e^{-2x} - 4u'e^{-2x} + 4ue^{-2x},$$

so

$$\begin{aligned} y'' + 4y' + 13y &= e^{-2x} [(u'' - 4u' + 4u) + 4(u' - 2u) + 13u] \\ &= e^{-2x} [u'' - (4 - 4)u' + (4 - 8 + 13)u] = e^{-2x}(u'' + 9u). \end{aligned}$$

Therefore $y = ue^{-2x}$ is a solution of (5.2.14) if and only if

$$u'' + 9u = 0.$$

From Example 5.1.2, the general solution of this equation is

$$u = c_1 \cos 3x + c_2 \sin 3x.$$

Therefore any function of the form

$$y = e^{-2x}(c_1 \cos 3x + c_2 \sin 3x) \quad (5.2.16)$$

is a solution of (5.2.14). Letting $c_1 = 1$ and $c_2 = 0$ yields the solution $y_1 = e^{-2x} \cos 3x$. Letting $c_1 = 0$ and $c_2 = 1$ yields the second solution $y_2 = e^{-2x} \sin 3x$. Since $y_2/y_1 = \tan 3x$ is nonconstant, 5.1.6 implies that $\{y_1, y_2\}$ is a fundamental set of solutions of (5.2.14), and (5.2.16) is the general solution.

Solution (b) Imposing the condition $y(0) = 2$ in (5.2.16) shows that $c_1 = 2$. Differentiating (5.2.16) yields

$$y' = -2e^{-2x}(c_1 \cos 3x + c_2 \sin 3x) + 3e^{-2x}(-c_1 \sin 3x + c_2 \cos 3x),$$

and imposing the initial condition $y'(0) = -3$ here yields $-3 = -2c_1 + 3c_2 = -4 + 3c_2$, so $c_2 = 1/3$. Therefore the solution of (5.2.15) is

$$y = e^{-2x}\left(2 \cos 3x + \frac{1}{3} \sin 3x\right).$$

Figure 5.2.3 is a graph of this function. ■

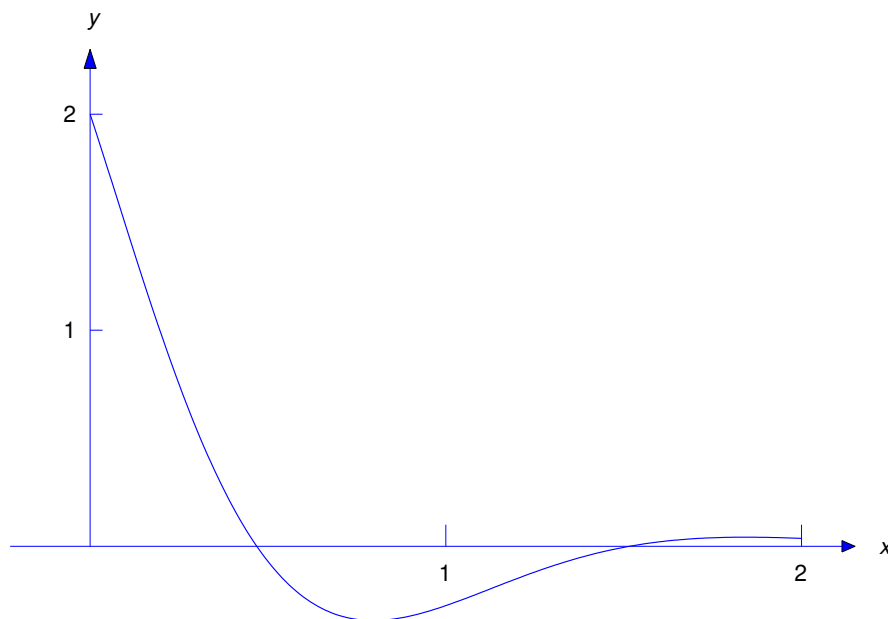


Figure 5.2.3 $y = e^{-2x}\left(2 \cos 3x + \frac{1}{3} \sin 3x\right)$

Section 5.3

Example 5.3.1

(a) Find the general solution of

$$y'' + y = 1. \quad (5.3.7)$$

(b) Solve the initial value problem

$$y'' + y = 1, \quad y(0) = 2, \quad y'(0) = 7. \quad (5.3.8)$$

Solution (a) We can apply Theorem 5.3.2 with $(a, b) = (-\infty, \infty)$, since the functions $p \equiv 0$, $q \equiv 1$, and $f \equiv 1$ in (5.3.7) are continuous on $(-\infty, \infty)$. By inspection we see that $y_p \equiv 1$ is a particular solution of (5.3.7). Since $y_1 = \cos x$ and $y_2 = \sin x$ form a fundamental set of solutions of the complementary equation $y'' + y = 0$, the general solution of (5.3.7) is

$$y = 1 + c_1 \cos x + c_2 \sin x. \quad (5.3.9)$$

Solution (b) Imposing the initial condition $y(0) = 2$ in (5.3.9) yields $2 = 1 + c_1$, so $c_1 = 1$. Differentiating (5.3.9) yields

$$y' = -c_1 \sin x + c_2 \cos x.$$

Imposing the initial condition $y'(0) = 7$ here yields $c_2 = 7$, so the solution of (5.3.8) is

$$y = 1 + \cos x + 7 \sin x.$$

Figure 5.3.1 is a graph of this function.

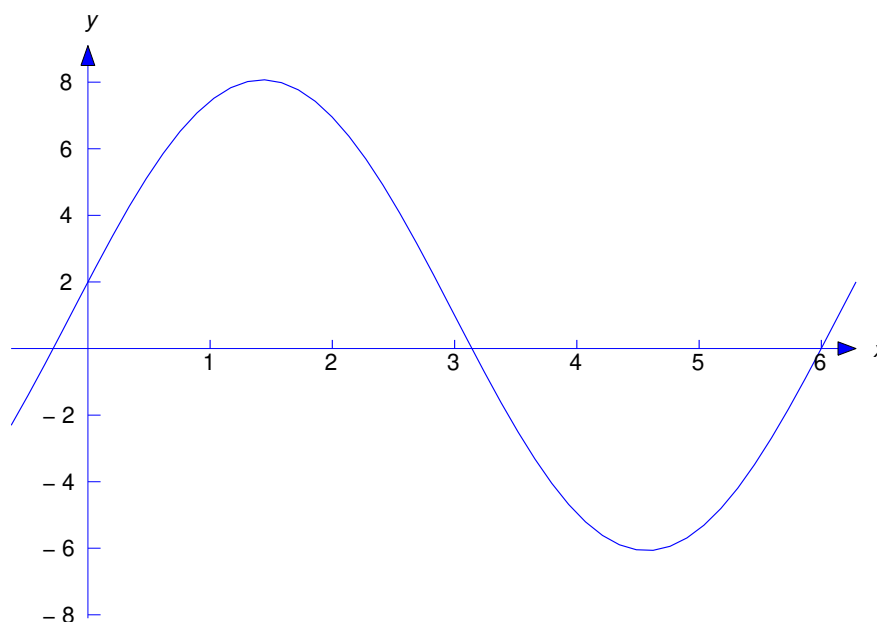


Figure 5.3.1 $y = 1 + \cos x + 7 \sin x$

Example 5.3.2

(a) Find the general solution of

$$y'' - 2y' + y = -3 - x + x^2. \quad (5.3.10)$$

(b) Solve the initial value problem

$$y'' - 2y' + y = -3 - x + x^2, \quad y(0) = -2, \quad y'(0) = 1. \quad (5.3.11)$$

Solution (a) The characteristic polynomial of the complementary equation

$$y'' - 2y' + y = 0$$

is $r^2 - 2r + 1 = (r - 1)^2$, so $y_1 = e^x$ and $y_2 = xe^x$ form a fundamental set of solutions of the complementary equation. To guess a form for a particular solution of (5.3.10), we note that substituting a second degree polynomial $y_p = A + Bx + Cx^2$ into the left side of (5.3.10) will produce another second degree polynomial with coefficients that depend upon A , B , and C . The trick is to choose A , B , and C so the polynomials on the two sides of (5.3.10) have the same coefficients; thus, if

$$y_p = A + Bx + Cx^2 \quad \text{then} \quad y'_p = B + 2Cx \quad \text{and} \quad y''_p = 2C,$$

so

$$\begin{aligned} y''_p - 2y'_p + y_p &= 2C - 2(B + 2Cx) + (A + Bx + Cx^2) \\ &= (2C - 2B + A) + (-4C + B)x + Cx^2 = -3 - x + x^2. \end{aligned}$$

Equating coefficients of like powers of x on the two sides of the last equality yields

$$\begin{aligned} C &= 1 \\ B - 4C &= -1 \\ A - 2B + 2C &= -3, \end{aligned}$$

so $C = 1$, $B = -1 + 4C = 3$, and $A = -3 - 2C + 2B = 1$. Therefore $y_p = 1 + 3x + x^2$ is a particular solution of (5.3.10) and Theorem 5.3.2 implies that

$$y = 1 + 3x + x^2 + e^x(c_1 + c_2x) \quad (5.3.12)$$

is the general solution of (5.3.10).

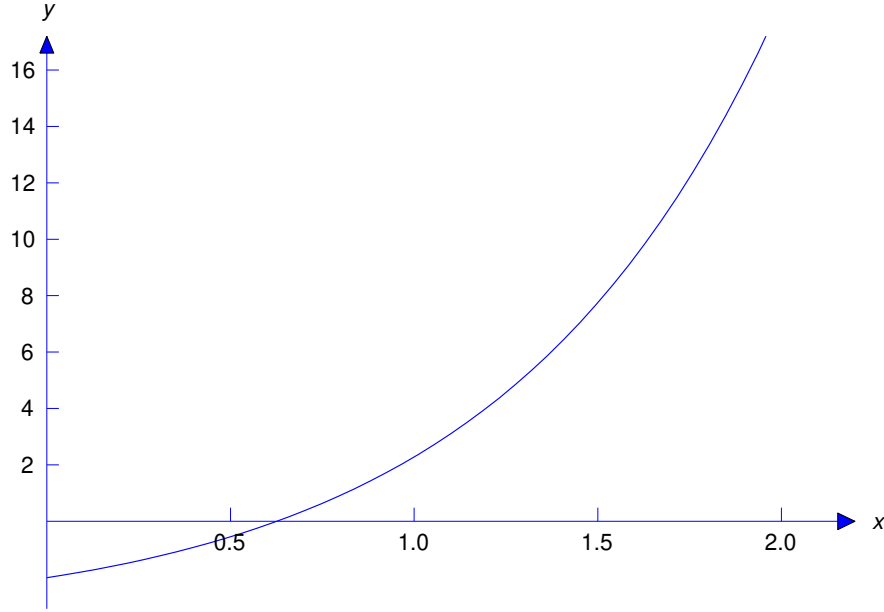
Solution (b) Imposing the initial condition $y(0) = -2$ in (5.3.12) yields $-2 = 1 + c_1$, so $c_1 = -3$. Differentiating (5.3.12) yields

$$y' = 3 + 2x + e^x(c_1 + c_2x) + c_2e^x,$$

and imposing the initial condition $y'(0) = 1$ here yields $1 = 3 + c_1 + c_2$, so $c_2 = 1$. Therefore the solution of (5.3.11) is

$$y = 1 + 3x + x^2 - e^x(3 - x).$$

Figure 5.3.2 is a graph of this solution.

Figure 5.3.2 $y = 1 + 3x + x^2 - e^x(3 - x)$

Example 5.3.3 Find the general solution of

$$x^2 y'' + xy' - 4y = 2x^4 \quad (5.3.13)$$

on $(-\infty, 0)$ and $(0, \infty)$.

Solution In Example 5.1.3, we verified that $y_1 = x^2$ and $y_2 = 1/x^2$ form a fundamental set of solutions of the complementary equation

$$x^2 y'' + xy' - 4y = 0$$

on $(-\infty, 0)$ and $(0, \infty)$. To find a particular solution of (5.3.13), we note that if $y_p = Ax^4$, where A is a constant then both sides of (5.3.13) will be constant multiples of x^4 and we may be able to choose A so the two sides are equal. This is true in this example, since if $y_p = Ax^4$ then

$$x^2 y_p'' + xy_p' - 4y_p = x^2(12Ax^2) + x(4Ax^3) - 4Ax^4 = 12Ax^4 = 2x^4$$

if $A = 1/6$; therefore, $y_p = x^4/6$ is a particular solution of (5.3.13) on $(-\infty, \infty)$. Theorem 5.3.2 implies that the general solution of (5.3.13) on $(-\infty, 0)$ and $(0, \infty)$ is

$$y = \frac{x^4}{6} + c_1 x^2 + \frac{c_2}{x^2}.$$

Example 5.3.4 The function $y_{p1} = x^4/15$ is a particular solution of

$$x^2 y'' + 4xy' + 2y = 2x^4 \quad (5.3.15)$$

on $(-\infty, \infty)$ and $y_{p2} = x^2/3$ is a particular solution of

$$x^2 y'' + 4xy' + 2y = 4x^2 \quad (5.3.16)$$

on $(-\infty, \infty)$. Use the principle of superposition to find a particular solution of

$$x^2 y'' + 4xy' + 2y = 2x^4 + 4x^2 \quad (5.3.17)$$

on $(-\infty, \infty)$.

Solution The right side $F(x) = 2x^4 + 4x^2$ in (5.3.17) is the sum of the right sides

$$F_1(x) = 2x^4 \quad \text{and} \quad F_2(x) = 4x^2$$

in (5.3.15) and (5.3.16). Therefore the principle of superposition implies that

$$y_p = y_{p_1} + y_{p_2} = \frac{x^4}{15} + \frac{x^2}{3}$$

is a particular solution of (5.3.17).

Section 5.4

Example 5.4.1 Find a particular solution of

$$y'' - 7y' + 12y = 4e^{2x}. \quad (5.4.2)$$

Then find the general solution.

Solution Substituting $y_p = Ae^{2x}$ for y in (5.4.2) will produce a constant multiple of Ae^{2x} on the left side of (5.4.2), so it may be possible to choose A so that y_p is a solution of (5.4.2). Let's try it; if $y_p = Ae^{2x}$ then

$$y_p'' - 7y_p' + 12y_p = 4Ae^{2x} - 14Ae^{2x} + 12Ae^{2x} = 2Ae^{2x} = 4e^{2x}$$

if $A = 2$. Therefore $y_p = 2e^{2x}$ is a particular solution of (5.4.2). To find the general solution, we note that the characteristic polynomial of the complementary equation

$$y'' - 7y' + 12y = 0 \quad (5.4.3)$$

is $p(r) = r^2 - 7r + 12 = (r - 3)(r - 4)$, so $\{e^{3x}, e^{4x}\}$ is a fundamental set of solutions of (5.4.3). Therefore the general solution of (5.4.2) is

$$y = 2e^{2x} + c_1e^{3x} + c_2e^{4x}.$$

Example 5.4.2 Find a particular solution of

$$y'' - 7y' + 12y = 5e^{4x}. \quad (5.4.4)$$

Then find the general solution.

Solution Fresh from our success in finding a particular solution of (5.4.2) — where we chose $y_p = Ae^{2x}$ because the right side of (5.4.2) is a constant multiple of e^{2x} — it may seem reasonable to try $y_p = Ae^{4x}$ as a particular solution of (5.4.4). However, this won't work, since we saw in Example 5.4.1 that e^{4x} is a solution of the complementary equation (5.4.3), so substituting $y_p = Ae^{4x}$ into the left side of (5.4.4) produces zero on the left, no matter how we choose A . To discover a suitable form for y_p , we use the same approach that we used in Section 5.2 to find a second solution of

$$ay'' + by' + cy = 0$$

in the case where the characteristic equation has a repeated real root: we look for solutions of (5.4.4) in the form $y = ue^{4x}$, where u is a function to be determined. Substituting

$$y = ue^{4x}, \quad y' = u'e^{4x} + 4ue^{4x}, \quad \text{and} \quad y'' = u''e^{4x} + 8u'e^{4x} + 16ue^{4x} \quad (5.4.5)$$

into (5.4.4) and canceling the common factor e^{4x} yields

$$(u'' + 8u' + 16u) - 7(u' + 4u) + 12u = 5,$$

or

$$u'' + u' = 5.$$

By inspection we see that $u_p = 5x$ is a particular solution of this equation, so $y_p = 5xe^{4x}$ is a particular solution of (5.4.4). Therefore

$$y = 5xe^{4x} + c_1e^{3x} + c_2e^{4x}$$

is the general solution.

Example 5.4.3 Find a particular solution of

$$y'' - 8y' + 16y = 2e^{4x}. \quad (5.4.6)$$

Solution Since the characteristic polynomial of the complementary equation

$$y'' - 8y' + 16y = 0 \quad (5.4.7)$$

is $p(r) = r^2 - 8r + 16 = (r - 4)^2$, both $y_1 = e^{4x}$ and $y_2 = xe^{4x}$ are solutions of (5.4.7). Therefore (5.4.6) does not have a solution of the form $y_p = Ae^{4x}$ or $y_p = Axe^{4x}$. As in Example 5.4.2, we look for solutions of (5.4.6) in the form $y = ue^{4x}$, where u is a function to be determined. Substituting from (5.4.5) into (5.4.6) and canceling the common factor e^{4x} yields

$$(u'' + 8u' + 16u) - 8(u' + 4u) + 16u = 2,$$

or

$$u'' = 2.$$

Integrating twice and taking the constants of integration to be zero shows that $u_p = x^2$ is a particular solution of this equation, so $y_p = x^2e^{4x}$ is a particular solution of (5.4.6). Therefore

$$y = e^{4x}(x^2 + c_1 + c_2x)$$

is the general solution. ■

Example 5.4.4 Find a particular solution of

$$y'' - 3y' + 2y = e^{3x}(-1 + 2x + x^2). \quad (5.4.10)$$

Solution Substituting

$$y = ue^{3x}, \quad y' = u'e^{3x} + 3ue^{3x}, \quad \text{and} \quad y'' = u''e^{3x} + 6u'e^{3x} + 9ue^{3x}$$

into (5.4.10) and canceling e^{3x} yields

$$(u'' + 6u' + 9u) - 3(u' + 3u) + 2u = -1 + 2x + x^2,$$

or

$$u'' + 3u' + 2u = -1 + 2x + x^2. \quad (5.4.11)$$

As in Example 5.3.2, in order to guess a form for a particular solution of (5.4.11), we note that substituting a second degree polynomial $u_p = A + Bx + Cx^2$ for u in the left side of (5.4.11) produces another second degree polynomial with coefficients that depend upon A , B , and C ; thus,

$$\text{if } u_p = A + Bx + Cx^2 \text{ then } u'_p = B + 2Cx \text{ and } u''_p = 2C.$$

If u_p is to satisfy (5.4.11), we must have

$$\begin{aligned} u''_p + 3u'_p + 2u_p &= 2C + 3(B + 2Cx) + 2(A + Bx + Cx^2) \\ &= (2C + 3B + 2A) + (6C + 2B)x + 2Cx^2 = -1 + 2x + x^2. \end{aligned}$$

Equating coefficients of like powers of x on the two sides of the last equality yields

$$\begin{aligned} 2C &= 1 \\ 2B + 6C &= 2 \\ 2A + 3B + 2C &= -1. \end{aligned}$$

Solving these equations for C , B , and A (in that order) yields $C = 1/2$, $B = -1/2$, $A = -1/4$. Therefore

$$u_p = -\frac{1}{4}(1 + 2x - 2x^2)$$

is a particular solution of (5.4.11), and

$$y_p = u_p e^{3x} = -\frac{e^{3x}}{4}(1 + 2x - 2x^2)$$

is a particular solution of (5.4.10).

Example 5.4.5 Find a particular solution of

$$y'' - 4y' + 3y = e^{3x}(6 + 8x + 12x^2). \quad (5.4.12)$$

Solution Substituting

$$y = ue^{3x}, \quad y' = u'e^{3x} + 3ue^{3x}, \quad \text{and } y'' = u''e^{3x} + 6u'e^{3x} + 9ue^{3x}$$

into (5.4.12) and canceling e^{3x} yields

$$(u'' + 6u' + 9u) - 4(u' + 3u) + 3u = 6 + 8x + 12x^2,$$

or

$$u'' + 2u' = 6 + 8x + 12x^2. \quad (5.4.13)$$

There's no u term in this equation, since e^{3x} is a solution of the complementary equation for (5.4.12). (See Exercise 5.4.30.) Therefore (5.4.13) does not have a particular solution of the

form $u_p = A + Bx + Cx^2$ that we used successfully in Example 5.4.4, since with this choice of u_p ,

$$u_p'' + 2u_p' = 2C + 2(B + 2Cx)$$

can't contain the last term ($12x^2$) on the right side of (5.4.13). Instead, let's try $u_p = Ax + Bx^2 + Cx^3$ on the grounds that

$$u_p' = A + 2Bx + 3Cx^2 \quad \text{and} \quad u_p'' = 2B + 6Cx$$

together contain all the powers of x that appear on the right side of (5.4.13).

Substituting these expressions in place of u' and u'' in (5.4.13) yields

$$(2B + 6Cx) + 2(A + 2Bx + 3Cx^2) = (2B + 2A) + (6C + 4B)x + 6Cx^2 = 6 + 8x + 12x^2.$$

Comparing coefficients of like powers of x on the two sides of the last equality shows that u_p satisfies (5.4.13) if

$$\begin{aligned} 6C &= 12 \\ 4B + 6C &= 8 \\ 2A + 2B &= 6. \end{aligned}$$

Solving these equations successively yields $C = 2$, $B = -1$, and $A = 4$. Therefore

$$u_p = x(4 - x + 2x^2)$$

is a particular solution of (5.4.13), and

$$y_p = u_p e^{3x} = x e^{3x} (4 - x + 2x^2)$$

is a particular solution of (5.4.12).

Example 5.4.6 Find a particular solution of

$$4y'' + 4y' + y = e^{-x/2}(-8 + 48x + 144x^2). \quad (5.4.14)$$

Solution Substituting

$$y = u e^{-x/2}, \quad y' = u' e^{-x/2} - \frac{1}{2} u e^{-x/2}, \quad \text{and} \quad y'' = u'' e^{-x/2} - u' e^{-x/2} + \frac{1}{4} u e^{-x/2}$$

into (5.4.14) and canceling $e^{-x/2}$ yields

$$4\left(u'' - u' + \frac{u}{4}\right) + 4\left(u' - \frac{u}{2}\right) + u = 4u'' = -8 + 48x + 144x^2,$$

or

$$u'' = -2 + 12x + 36x^2, \quad (5.4.15)$$

which does not contain u or u' because $e^{-x/2}$ and $x e^{-x/2}$ are both solutions of the complementary equation. (See Exercise 5.4.30.) To obtain a particular solution of (5.4.15) we integrate twice, taking the constants of integration to be zero; thus,

$$u_p' = -2x + 6x^2 + 12x^3 \quad \text{and} \quad u_p = -x^2 + 2x^3 + 3x^4 = x^2(-1 + 2x + 3x^2).$$

Therefore

$$y_p = u_p e^{-x/2} = x^2 e^{-x/2} (-1 + 2x + 3x^2)$$

is a particular solution of (5.4.14).

Example 5.4.7 Find a particular solution of

$$y'' - 7y' + 12y = 4e^{2x} + 5e^{4x}. \quad (5.4.17)$$

Solution In Example 5.4.1 we found that $y_{p_1} = 2e^{2x}$ is a particular solution of

$$y'' - 7y' + 12y = 4e^{2x},$$

and in Example 5.4.2 we found that $y_{p_2} = 5xe^{4x}$ is a particular solution of

$$y'' - 7y' + 12y = 5e^{4x}.$$

Therefore the principle of superposition implies that $y_p = 2e^{2x} + 5xe^{4x}$ is a particular solution of (5.4.17).

Section 5.5

Example 5.5.1 Find a particular solution of

$$y'' - 2y' + y = 5 \cos 2x + 10 \sin 2x. \quad (5.5.4)$$

Solution In (5.5.4) the coefficients of $\cos 2x$ and $\sin 2x$ are both zero degree polynomials (constants). Therefore Theorem 5.5.1 implies that (5.5.4) has a particular solution

$$y_p = A \cos 2x + B \sin 2x.$$

Since

$$y'_p = -2A \sin 2x + 2B \cos 2x \quad \text{and} \quad y''_p = -4(A \cos 2x + B \sin 2x),$$

replacing y by y_p in (5.5.4) yields

$$\begin{aligned} y''_p - 2y'_p + y_p &= -4(A \cos 2x + B \sin 2x) - 4(-A \sin 2x + B \cos 2x) \\ &\quad + (A \cos 2x + B \sin 2x) \\ &= (-3A - 4B) \cos 2x + (4A - 3B) \sin 2x. \end{aligned}$$

Equating the coefficients of $\cos 2x$ and $\sin 2x$ here with the corresponding coefficients on the right side of (5.5.4) shows that y_p is a solution of (5.5.4) if

$$\begin{aligned} -3A - 4B &= 5 \\ 4A - 3B &= 10. \end{aligned}$$

Solving these equations yields $A = 1$, $B = -2$. Therefore

$$y_p = \cos 2x - 2 \sin 2x$$

is a particular solution of (5.5.4).

Example 5.5.2 Find a particular solution of

$$y'' + 4y = 8 \cos 2x + 12 \sin 2x. \quad (5.5.5)$$

Solution The procedure used in Example 5.5.1 doesn't work here; substituting $y_p = A \cos 2x + B \sin 2x$ for y in (5.5.5) yields

$$y_p'' + 4y_p = -4(A \cos 2x + B \sin 2x) + 4(A \cos 2x + B \sin 2x) = 0$$

for any choice of A and B , since $\cos 2x$ and $\sin 2x$ are both solutions of the complementary equation for (5.5.5). We're dealing with the second case mentioned in Theorem 5.5.1, and should therefore try a particular solution of the form

$$y_p = x(A \cos 2x + B \sin 2x). \quad (5.5.6)$$

Then

$$y_p' = A \cos 2x + B \sin 2x + 2x(-A \sin 2x + B \cos 2x)$$

and

$$\begin{aligned} y_p'' &= -4A \sin 2x + 4B \cos 2x - 4x(A \cos 2x + B \sin 2x) \\ &= -4A \sin 2x + 4B \cos 2x - 4y_p \text{ (see (5.5.6))}, \end{aligned}$$

so

$$y_p'' + 4y_p = -4A \sin 2x + 4B \cos 2x.$$

Therefore y_p is a solution of (5.5.5) if

$$-4A \sin 2x + 4B \cos 2x = 8 \cos 2x + 12 \sin 2x,$$

which holds if $A = -3$ and $B = 2$. Therefore

$$y_p = -x(3 \cos 2x - 2 \sin 2x)$$

is a particular solution of (5.5.5).

Example 5.5.3 Find a particular solution of

$$y'' + 3y' + 2y = (16 + 20x) \cos x + 10 \sin x. \quad (5.5.7)$$

Solution The coefficients of $\cos x$ and $\sin x$ in (5.5.7) are polynomials of degree one and zero, respectively. Therefore Theorem 5.5.1 tells us to look for a particular solution of (5.5.7) of the form

$$y_p = (A_0 + A_1x) \cos x + (B_0 + B_1x) \sin x. \quad (5.5.8)$$

Then

$$y_p' = (A_1 + B_0 + B_1x) \cos x + (B_1 - A_0 - A_1x) \sin x \quad (5.5.9)$$

and

$$y_p'' = (2B_1 - A_0 - A_1x) \cos x - (2A_1 + B_0 + B_1x) \sin x, \quad (5.5.10)$$

so

$$\begin{aligned} y_p'' + 3y_p' + 2y_p &= [A_0 + 3A_1 + 3B_0 + 2B_1 + (A_1 + 3B_1)x] \cos x \\ &\quad + [B_0 + 3B_1 - 3A_0 - 2A_1 + (B_1 - 3A_1)x] \sin x. \end{aligned} \quad (5.5.11)$$

Comparing the coefficients of $x \cos x$, $x \sin x$, $\cos x$, and $\sin x$ here with the corresponding coefficients in (5.5.7) shows that y_p is a solution of (5.5.7) if

$$\begin{aligned} A_1 + 3B_1 &= 20 \\ -3A_1 + B_1 &= 0 \\ A_0 + 3B_0 + 3A_1 + 2B_1 &= 16 \\ -3A_0 + B_0 - 2A_1 + 3B_1 &= 10. \end{aligned}$$

Solving the first two equations yields $A_1 = 2$, $B_1 = 6$. Substituting these into the last two equations yields

$$\begin{aligned} A_0 + 3B_0 &= 16 - 3A_1 - 2B_1 = -2 \\ -3A_0 + B_0 &= 10 + 2A_1 - 3B_1 = -4. \end{aligned}$$

Solving these equations yields $A_0 = 1$, $B_0 = -1$. Substituting $A_0 = 1$, $A_1 = 2$, $B_0 = -1$, $B_1 = 6$ into (5.5.8) shows that

$$y_p = (1 + 2x) \cos x - (1 - 6x) \sin x$$

is a particular solution of (5.5.7).

Example 5.5.4 Find a particular solution of

$$y'' + y = (8 - 4x) \cos x - (8 + 8x) \sin x. \quad (5.5.12)$$

Solution According to Theorem 5.5.1, we should look for a particular solution of the form

$$y_p = (A_0x + A_1x^2) \cos x + (B_0x + B_1x^2) \sin x, \quad (5.5.13)$$

since $\cos x$ and $\sin x$ are solutions of the complementary equation. However, let's try

$$y_p = (A_0 + A_1x) \cos x + (B_0 + B_1x) \sin x \quad (5.5.14)$$

first, so you can see why it doesn't work. From (5.5.10),

$$y_p'' = (2B_1 - A_0 - A_1x) \cos x - (2A_1 + B_0 + B_1x) \sin x,$$

which together with (5.5.14) implies that

$$y_p'' + y_p = 2B_1 \cos x - 2A_1 \sin x.$$

Since the right side of this equation does not contain $x \cos x$ or $x \sin x$, (5.5.14) can't satisfy (5.5.12) no matter how we choose A_0 , A_1 , B_0 , and B_1 .

Now let y_p be as in (5.5.13). Then

$$\begin{aligned} y_p' &= [A_0 + (2A_1 + B_0)x + B_1x^2] \cos x \\ &\quad + [B_0 + (2B_1 - A_0)x - A_1x^2] \sin x \end{aligned}$$

and

$$\begin{aligned} y_p'' &= [2A_1 + 2B_0 - (A_0 - 4B_1)x - A_1x^2] \cos x \\ &\quad + [2B_1 - 2A_0 - (B_0 + 4A_1)x - B_1x^2] \sin x, \end{aligned}$$

so

$$y_p'' + y_p = (2A_1 + 2B_0 + 4B_1x) \cos x + (2B_1 - 2A_0 - 4A_1x) \sin x.$$

Comparing the coefficients of $\cos x$ and $\sin x$ here with the corresponding coefficients in (5.5.12) shows that y_p is a solution of (5.5.12) if

$$\begin{aligned} 4B_1 &= -4 \\ -4A_1 &= -8 \\ 2B_0 + 2A_1 &= 8 \\ -2A_0 + 2B_1 &= -8. \end{aligned}$$

The solution of this system is $A_1 = 2$, $B_1 = -1$, $A_0 = 3$, $B_0 = 2$. Therefore

$$y_p = x [(3 + 2x) \cos x + (2 - x) \sin x]$$

is a particular solution of (5.5.12).

Example 5.5.5 Find a particular solution of

$$y'' - 3y' + 2y = e^{-2x} [2 \cos 3x - (34 - 150x) \sin 3x]. \quad (5.5.16)$$

Solution Let $y = ue^{-2x}$. Then

$$\begin{aligned} y'' - 3y' + 2y &= e^{-2x} [(u'' - 4u' + 4u) - 3(u' - 2u) + 2u] \\ &= e^{-2x} (u'' - 7u' + 12u) \\ &= e^{-2x} [2 \cos 3x - (34 - 150x) \sin 3x] \end{aligned}$$

if

$$u'' - 7u' + 12u = 2 \cos 3x - (34 - 150x) \sin 3x. \quad (5.5.17)$$

Since $\cos 3x$ and $\sin 3x$ aren't solutions of the complementary equation

$$u'' - 7u' + 12u = 0,$$

Theorem 5.5.1 tells us to look for a particular solution of (5.5.17) of the form

$$u_p = (A_0 + A_1x) \cos 3x + (B_0 + B_1x) \sin 3x. \quad (5.5.18)$$

Then

$$u'_p = (A_1 + 3B_0 + 3B_1x) \cos 3x + (B_1 - 3A_0 - 3A_1x) \sin 3x$$

and

$$u''_p = (-9A_0 + 6B_1 - 9A_1x) \cos 3x - (9B_0 + 6A_1 + 9B_1x) \sin 3x,$$

so

$$\begin{aligned} u''_p - 7u'_p + 12u_p &= [3A_0 - 21B_0 - 7A_1 + 6B_1 + (3A_1 - 21B_1)x] \cos 3x \\ &\quad + [21A_0 + 3B_0 - 6A_1 - 7B_1 + (21A_1 + 3B_1)x] \sin 3x. \end{aligned}$$

Comparing the coefficients of $x \cos 3x$, $x \sin 3x$, $\cos 3x$, and $\sin 3x$ here with the corresponding coefficients on the right side of (5.5.17) shows that u_p is a solution of (5.5.17) if

$$\begin{aligned} 3A_1 - 21B_1 &= 0 \\ 21A_1 + 3B_1 &= 150 \\ 3A_0 - 21B_0 - 7A_1 + 6B_1 &= 2 \\ 21A_0 + 3B_0 - 6A_1 - 7B_1 &= -34. \end{aligned} \quad (5.5.19)$$

Solving the first two equations yields $A_1 = 7$, $B_1 = 1$. Substituting these values into the last two equations of (5.5.19) yields

$$\begin{aligned} 3A_0 - 21B_0 &= 2 + 7A_1 - 6B_1 = 45 \\ 21A_0 + 3B_0 &= -34 + 6A_1 + 7B_1 = 15. \end{aligned}$$

Solving this system yields $A_0 = 1$, $B_0 = -2$. Substituting $A_0 = 1$, $A_1 = 7$, $B_0 = -2$, and $B_1 = 1$ into (5.5.18) shows that

$$u_p = (1 + 7x) \cos 3x - (2 - x) \sin 3x$$

is a particular solution of (5.5.17). Therefore

$$y_p = e^{-2x} [(1 + 7x) \cos 3x - (2 - x) \sin 3x]$$

is a particular solution of (5.5.16).

Example 5.5.6 Find a particular solution of

$$y'' + 2y' + 5y = e^{-x} [(6 - 16x) \cos 2x - (8 + 8x) \sin 2x]. \quad (5.5.20)$$

Solution Let $y = ue^{-x}$. Then

$$\begin{aligned} y'' + 2y' + 5y &= e^{-x} [(u'' - 2u' + u) + 2(u' - u) + 5u] \\ &= e^{-x} (u'' + 4u) \\ &= e^{-x} [(6 - 16x) \cos 2x - (8 + 8x) \sin 2x] \end{aligned}$$

if

$$u'' + 4u = (6 - 16x) \cos 2x - (8 + 8x) \sin 2x. \quad (5.5.21)$$

Since $\cos 2x$ and $\sin 2x$ are solutions of the complementary equation

$$u'' + 4u = 0,$$

Theorem 5.5.1 tells us to look for a particular solution of (5.5.21) of the form

$$u_p = (A_0x + A_1x^2) \cos 2x + (B_0x + B_1x^2) \sin 2x.$$

Then

$$\begin{aligned} u'_p &= [A_0 + (2A_1 + 2B_0)x + 2B_1x^2] \cos 2x \\ &\quad + [B_0 + (2B_1 - 2A_0)x - 2A_1x^2] \sin 2x \end{aligned}$$

and

$$\begin{aligned} u''_p &= [2A_1 + 4B_0 - (4A_0 - 8B_1)x - 4A_1x^2] \cos 2x \\ &\quad + [2B_1 - 4A_0 - (4B_0 + 8A_1)x - 4B_1x^2] \sin 2x, \end{aligned}$$

so

$$u''_p + 4u_p = (2A_1 + 4B_0 + 8B_1x) \cos 2x + (2B_1 - 4A_0 - 8A_1x) \sin 2x.$$

Equating the coefficients of $x \cos 2x$, $x \sin 2x$, $\cos 2x$, and $\sin 2x$ here with the corresponding coefficients on the right side of (5.5.21) shows that u_p is a solution of (5.5.21) if

$$\begin{aligned} 8B_1 &= -16 \\ -8A_1 &= -8 \\ 4B_0 + 2A_1 &= 6 \\ -4A_0 + 2B_1 &= -8. \end{aligned} \quad (5.5.22)$$

The solution of this system is $A_1 = 1$, $B_1 = -2$, $B_0 = 1$, $A_0 = 1$. Therefore

$$u_p = x[(1 + x) \cos 2x + (1 - 2x) \sin 2x]$$

is a particular solution of (5.5.21), and

$$y_p = xe^{-x} [(1 + x) \cos 2x + (1 - 2x) \sin 2x]$$

is a particular solution of (5.5.20). ■

You can also find a particular solution of (5.5.20) by substituting

$$y_p = xe^{-x} [(A_0 + A_1x) \cos 2x + (B_0 + B_1x) \sin 2x]$$

for y in (5.5.20) and equating the coefficients of $xe^{-x} \cos 2x$, $xe^{-x} \sin 2x$, $e^{-x} \cos 2x$, and $e^{-x} \sin 2x$ in the resulting expression for

$$y_p'' + 2y_p' + 5y_p$$

with the corresponding coefficients on the right side of (5.5.20). (See Exercise 5.5.38). This leads to the same system (5.5.22) of equations for A_0 , A_1 , B_0 , and B_1 that we obtained in Example 5.5.6. However, if you try this approach you'll see that deriving (5.5.22) this way is much more tedious than the way we did it in Example 5.5.6.

Section 5.6

Example 5.6.1

(a) Find the general solution of

$$xy'' - (2x + 1)y' + (x + 1)y = x^2, \quad (5.6.6)$$

given that $y_1 = e^x$ is a solution of the complementary equation

$$xy'' - (2x + 1)y' + (x + 1)y = 0. \quad (5.6.7)$$

(b) As a byproduct of (a), find a fundamental set of solutions of (5.6.7).

Solution (a) If $y = ue^x$, then $y' = u'e^x + ue^x$ and $y'' = u''e^x + 2u'e^x + ue^x$, so

$$\begin{aligned} xy'' - (2x + 1)y' + (x + 1)y &= x(u''e^x + 2u'e^x + ue^x) \\ &\quad - (2x + 1)(u'e^x + ue^x) + (x + 1)ue^x \\ &= (xu'' - u')e^x. \end{aligned}$$

Therefore $y = ue^x$ is a solution of (5.6.6) if and only if

$$(xu'' - u')e^x = x^2,$$

which is a first order equation in u' . We rewrite it as

$$u'' - \frac{u'}{x} = xe^{-x}. \quad (5.6.8)$$

To focus on how we apply variation of parameters to this equation, we temporarily write $z = u'$, so that (5.6.8) becomes

$$z' - \frac{z}{x} = xe^{-x}. \quad (5.6.9)$$

We leave it to you to show (by separation of variables) that $z_1 = x$ is a solution of the complementary equation

$$z' - \frac{z}{x} = 0$$

for (5.6.9). By applying variation of parameters as in Section 2.1, we can now see that every solution of (5.6.9) is of the form

$$z = vx \quad \text{where} \quad v'x = xe^{-x}, \quad \text{so} \quad v' = e^{-x} \quad \text{and} \quad v = -e^{-x} + C_1.$$

Since $u' = z = vx$, u is a solution of (5.6.8) if and only if

$$u' = vx = -xe^{-x} + C_1x.$$

Integrating this yields

$$u = (x + 1)e^{-x} + \frac{C_1}{2}x^2 + C_2.$$

Therefore the general solution of (5.6.6) is

$$y = ue^x = x + 1 + \frac{C_1}{2}x^2e^x + C_2e^x. \quad (5.6.10)$$

Solution (b) By letting $C_1 = C_2 = 0$ in (5.6.10), we see that $y_{p_1} = x + 1$ is a solution of (5.6.6). By letting $C_1 = 2$ and $C_2 = 0$, we see that $y_{p_2} = x + 1 + x^2e^x$ is also a solution of (5.6.6). Since the difference of two solutions of (5.6.6) is a solution of (5.6.7), $y_2 = y_{p_2} - y_{p_1} = x^2e^x$ is a solution of (5.6.7). Since y_2/y_1 is nonconstant and we already know that $y_1 = e^x$ is a solution of (5.6.7), Theorem 5.1.6 implies that $\{e^x, x^2e^x\}$ is a fundamental set of solutions of (5.6.7). ■

Although (5.6.10) is a correct form for the general solution of (5.6.6), it's silly to leave the arbitrary coefficient of x^2e^x as $C_1/2$ where C_1 is an arbitrary constant. Moreover, it's sensible to make the subscripts of the coefficients of $y_1 = e^x$ and $y_2 = x^2e^x$ consistent with the subscripts of the functions themselves. Therefore we rewrite (5.6.10) as

$$y = x + 1 + c_1e^x + c_2x^2e^x$$

by simply renaming the arbitrary constants. We'll also do this in the next two examples, and in the answers to the exercises.

Example 5.6.2

(a) Find the general solution of

$$x^2y'' + xy' - y = x^2 + 1,$$

given that $y_1 = x$ is a solution of the complementary equation

$$x^2y'' + xy' - y = 0. \quad (5.6.11)$$

As a byproduct of this result, find a fundamental set of solutions of (5.6.11).

(b) Solve the initial value problem

$$x^2y'' + xy' - y = x^2 + 1, \quad y(1) = 2, \quad y'(1) = -3. \quad (5.6.12)$$

Solution (a) If $y = ux$, then $y' = u'x + u$ and $y'' = u''x + 2u'$, so

$$\begin{aligned} x^2y'' + xy' - y &= x^2(u''x + 2u') + x(u'x + u) - ux \\ &= x^3u'' + 3x^2u'. \end{aligned}$$

Therefore $y = ux$ is a solution of (5.6.12) if and only if

$$x^3u'' + 3x^2u' = x^2 + 1,$$

which is a first order equation in u' . We rewrite it as

$$u'' + \frac{3}{x}u' = \frac{1}{x} + \frac{1}{x^3}. \quad (5.6.13)$$

To focus on how we apply variation of parameters to this equation, we temporarily write $z = u'$, so that (5.6.13) becomes

$$z' + \frac{3}{x}z = \frac{1}{x} + \frac{1}{x^3}. \quad (5.6.14)$$

We leave it to you to show by separation of variables that $z_1 = 1/x^3$ is a solution of the complementary equation

$$z' + \frac{3}{x}z = 0$$

for (5.6.14). By variation of parameters, every solution of (5.6.14) is of the form

$$z = \frac{v}{x^3} \quad \text{where} \quad \frac{v'}{x^3} = \frac{1}{x} + \frac{1}{x^3}, \quad \text{so} \quad v' = x^2 + 1 \quad \text{and} \quad v = \frac{x^3}{3} + x + C_1.$$

Since $u' = z = v/x^3$, u is a solution of (5.6.13) if and only if

$$u' = \frac{v}{x^3} = \frac{1}{3} + \frac{1}{x^2} + \frac{C_1}{x^3}.$$

Integrating this yields

$$u = \frac{x}{3} - \frac{1}{x} - \frac{C_1}{2x^2} + C_2.$$

Therefore the general solution of (5.6.12) is

$$y = ux = \frac{x^2}{3} - 1 - \frac{C_1}{2x} + C_2x. \quad (5.6.15)$$

Reasoning as in the solution of Example 5.6.1(a), we conclude that $y_1 = x$ and $y_2 = 1/x$ form a fundamental set of solutions for (5.6.11).

As we explained above, we rename the constants in (5.6.15) and rewrite it as

$$y = \frac{x^2}{3} - 1 + c_1x + \frac{c_2}{x}. \quad (5.6.16)$$

Solution (b) Differentiating (5.6.16) yields

$$y' = \frac{2x}{3} + c_1 - \frac{c_2}{x^2}. \quad (5.6.17)$$

Setting $x = 1$ in (5.6.16) and (5.6.17) and imposing the initial conditions $y(1) = 2$ and $y'(1) = -3$ yields

$$\begin{aligned} c_1 + c_2 &= \frac{8}{3} \\ c_1 - c_2 &= -\frac{11}{3}. \end{aligned}$$

Solving these equations yields $c_1 = -1/2$, $c_2 = 19/6$. Therefore the solution of (5.6.12) is

$$y = \frac{x^2}{3} - 1 - \frac{x}{2} + \frac{19}{6x}.$$

Example 5.6.3 Find the general solution and a fundamental set of solutions of

$$x^2y'' - 3xy' + 3y = 0, \quad (5.6.18)$$

given that $y_1 = x$ is a solution.

Solution If $y = ux$ then $y' = u'x + u$ and $y'' = u''x + 2u'$, so

$$\begin{aligned} x^2y'' - 3xy' + 3y &= x^2(u''x + 2u') - 3x(u'x + u) + 3ux \\ &= x^3u'' - x^2u'. \end{aligned}$$

Therefore $y = ux$ is a solution of (5.6.18) if and only if

$$x^3u'' - x^2u' = 0.$$

Separating the variables u' and x yields

$$\frac{u''}{u'} = \frac{1}{x},$$

so

$$\ln |u'| = \ln |x| + k, \quad \text{or, equivalently,} \quad u' = C_1x.$$

Therefore

$$u = \frac{C_1}{2}x^2 + C_2,$$

so the general solution of (5.6.18) is

$$y = ux = \frac{C_1}{2}x^3 + C_2x,$$

which we rewrite as

$$y = c_1x + c_2x^3.$$

Therefore $\{x, x^3\}$ is a fundamental set of solutions of (5.6.18).

Section 5.7

Example 5.7.1 Find a particular solution y_p of

$$x^2y'' - 2xy' + 2y = x^{9/2}, \quad (5.7.15)$$

given that $y_1 = x$ and $y_2 = x^2$ are solutions of the complementary equation

$$x^2y'' - 2xy' + 2y = 0.$$

Then find the general solution of (5.7.15).

Solution We set

$$y_p = u_1x + u_2x^2,$$

where

$$\begin{aligned}u_1'x + u_2'x^2 &= 0 \\u_1' + 2u_2'x &= \frac{x^{9/2}}{x^2} = x^{5/2}.\end{aligned}$$

From the first equation, $u_1' = -u_2'x$. Substituting this into the second equation yields $u_2'x = x^{5/2}$, so $u_2' = x^{3/2}$ and therefore $u_1' = -u_2'x = -x^{5/2}$. Integrating and taking the constants of integration to be zero yields

$$u_1 = -\frac{2}{7}x^{7/2} \quad \text{and} \quad u_2 = \frac{2}{5}x^{5/2}.$$

Therefore

$$y_p = u_1x + u_2x^2 = -\frac{2}{7}x^{7/2}x + \frac{2}{5}x^{5/2}x^2 = \frac{4}{35}x^{9/2},$$

and the general solution of (5.7.15) is

$$y = \frac{4}{35}x^{9/2} + c_1x + c_2x^2.$$

Example 5.7.2 Find a particular solution y_p of

$$(x-1)y'' - xy' + y = (x-1)^2, \quad (5.7.16)$$

given that $y_1 = x$ and $y_2 = e^x$ are solutions of the complementary equation

$$(x-1)y'' - xy' + y = 0.$$

Then find the general solution of (5.7.16).

Solution We set

$$y_p = u_1x + u_2e^x,$$

where

$$\begin{aligned}u_1'x + u_2'e^x &= 0 \\u_1' + u_2'e^x &= \frac{(x-1)^2}{x-1} = x-1.\end{aligned}$$

Subtracting the first equation from the second yields $-u_1'(x-1) = x-1$, so $u_1' = -1$. From this and the first equation, $u_2' = -xe^{-x}u_1' = xe^{-x}$. Integrating and taking the constants of integration to be zero yields

$$u_1 = -x \quad \text{and} \quad u_2 = -(x+1)e^{-x}.$$

Therefore

$$y_p = u_1x + u_2e^x = (-x)x + (-(x+1)e^{-x})e^x = -x^2 - x - 1,$$

so the general solution of (5.7.16) is

$$y = y_p + c_1x + c_2e^x = -x^2 - x - 1 + c_1x + c_2e^x = -x^2 - 1 + (c_1 - 1)x + c_2e^x. \quad (5.7.17)$$

However, since c_1 is an arbitrary constant, so is $c_1 - 1$; therefore, we improve the appearance of this result by renaming the constant and writing the general solution as

$$y = -x^2 - 1 + c_1x + c_2e^x. \quad \blacksquare \quad (5.7.18)$$

Example 5.7.3 Find a particular solution of

$$y'' + 3y' + 2y = \frac{1}{1 + e^x}. \quad (5.7.19)$$

Then find the general solution.

Solution

The characteristic polynomial of the complementary equation

$$y'' + 3y' + 2y = 0 \quad (5.7.20)$$

is $p(r) = r^2 + 3r + 2 = (r + 1)(r + 2)$, so $y_1 = e^{-x}$ and $y_2 = e^{-2x}$ form a fundamental set of solutions of (5.7.20). We look for a particular solution of (5.7.19) in the form

$$y_p = u_1 e^{-x} + u_2 e^{-2x},$$

where

$$\begin{aligned} u_1' e^{-x} + u_2' e^{-2x} &= 0 \\ -u_1' e^{-x} - 2u_2' e^{-2x} &= \frac{1}{1 + e^x}. \end{aligned}$$

Adding these two equations yields

$$-u_2' e^{-2x} = \frac{1}{1 + e^x}, \quad \text{so} \quad u_2' = -\frac{e^{2x}}{1 + e^x}.$$

From the first equation,

$$u_1' = -u_2' e^{-x} = \frac{e^x}{1 + e^x}.$$

Integrating by means of the substitution $v = e^x$ and taking the constants of integration to be zero yields

$$u_1 = \int \frac{e^x}{1 + e^x} dx = \int \frac{dv}{1 + v} = \ln(1 + v) = \ln(1 + e^x)$$

and

$$\begin{aligned} u_2 &= - \int \frac{e^{2x}}{1 + e^x} dx = - \int \frac{v}{1 + v} dv = \int \left[\frac{1}{1 + v} - 1 \right] dv \\ &= \ln(1 + v) - v = \ln(1 + e^x) - e^x. \end{aligned}$$

Therefore

$$\begin{aligned} y_p &= u_1 e^{-x} + u_2 e^{-2x} \\ &= [\ln(1 + e^x)] e^{-x} + [\ln(1 + e^x) - e^x] e^{-2x}, \end{aligned}$$

so

$$y_p = (e^{-x} + e^{-2x}) \ln(1 + e^x) - e^{-x}.$$

Since the last term on the right satisfies the complementary equation, we drop it and redefine

$$y_p = (e^{-x} + e^{-2x}) \ln(1 + e^x).$$

The general solution of (5.7.19) is

$$y = y_p + c_1 e^{-x} + c_2 e^{-2x} = (e^{-x} + e^{-2x}) \ln(1 + e^x) + c_1 e^{-x} + c_2 e^{-2x}.$$

Example 5.7.4 Solve the initial value problem

$$(x^2 - 1)y'' + 4xy' + 2y = \frac{2}{x+1}, \quad y(0) = -1, \quad y'(0) = -5, \quad (5.7.21)$$

given that

$$y_1 = \frac{1}{x-1} \quad \text{and} \quad y_2 = \frac{1}{x+1}$$

are solutions of the complementary equation

$$(x^2 - 1)y'' + 4xy' + 2y = 0.$$

Solution We first use variation of parameters to find a particular solution of

$$(x^2 - 1)y'' + 4xy' + 2y = \frac{2}{x+1}$$

on $(-1, 1)$ in the form

$$y_p = \frac{u_1}{x-1} + \frac{u_2}{x+1},$$

where

$$\begin{aligned} \frac{u_1'}{x-1} + \frac{u_2'}{x+1} &= 0 \\ -\frac{u_1'}{(x-1)^2} - \frac{u_2'}{(x+1)^2} &= \frac{2}{(x+1)(x^2-1)}. \end{aligned} \quad (5.7.22)$$

Multiplying the first equation by $1/(x-1)$ and adding the result to the second equation yields

$$\left[\frac{1}{x^2-1} - \frac{1}{(x+1)^2} \right] u_2' = \frac{2}{(x+1)(x^2-1)}. \quad (5.7.23)$$

Since

$$\left[\frac{1}{x^2-1} - \frac{1}{(x+1)^2} \right] = \frac{(x+1) - (x-1)}{(x+1)(x^2-1)} = \frac{2}{(x+1)(x^2-1)},$$

(5.7.23) implies that $u_2' = 1$. From (5.7.22),

$$u_1' = -\frac{x-1}{x+1} u_2' = -\frac{x-1}{x+1}.$$

Integrating and taking the constants of integration to be zero yields

$$\begin{aligned} u_1 &= -\int \frac{x-1}{x+1} dx = -\int \frac{x+1-2}{x+1} dx \\ &= \int \left[\frac{2}{x+1} - 1 \right] dx = 2 \ln(x+1) - x \end{aligned}$$

and

$$u_2 = \int dx = x.$$

Therefore

$$\begin{aligned} y_p &= \frac{u_1}{x-1} + \frac{u_2}{x+1} = [2 \ln(x+1) - x] \frac{1}{x-1} + x \frac{1}{x+1} \\ &= \frac{2 \ln(x+1)}{x-1} + x \left[\frac{1}{x+1} - \frac{1}{x-1} \right] = \frac{2 \ln(x+1)}{x-1} - \frac{2x}{(x+1)(x-1)}. \end{aligned}$$

However, since

$$\frac{2x}{(x+1)(x-1)} = \left[\frac{1}{x+1} + \frac{1}{x-1} \right]$$

is a solution of the complementary equation, we redefine

$$y_p = \frac{2 \ln(x+1)}{x-1}.$$

Therefore the general solution of the differential equation in (5.7.21) is

$$y = \frac{2 \ln(x+1)}{x-1} + \frac{c_1}{x-1} + \frac{c_2}{x+1}. \quad (5.7.24)$$

Differentiating this yields

$$y' = \frac{2}{x^2-1} - \frac{2 \ln(x+1)}{(x-1)^2} - \frac{c_1}{(x-1)^2} - \frac{c_2}{(x+1)^2}.$$

Setting $x = 0$ in the last two equations and imposing the initial conditions $y(0) = -1$ and $y'(0) = -5$ yields the system

$$\begin{aligned} -c_1 + c_2 &= -1 \\ -2 - c_1 - c_2 &= -5. \end{aligned}$$

The solution of this system is $c_1 = 2$, $c_2 = 1$. Substituting these into (5.7.24) yields

$$\begin{aligned} y &= \frac{2 \ln(x+1)}{x-1} + \frac{2}{x-1} + \frac{1}{x+1} \\ &= \frac{2 \ln(x+1)}{x-1} + \frac{3x+1}{x^2-1} \end{aligned}$$

as the solution of (5.7.21). Figure 5.7.1 is a graph of the solution.

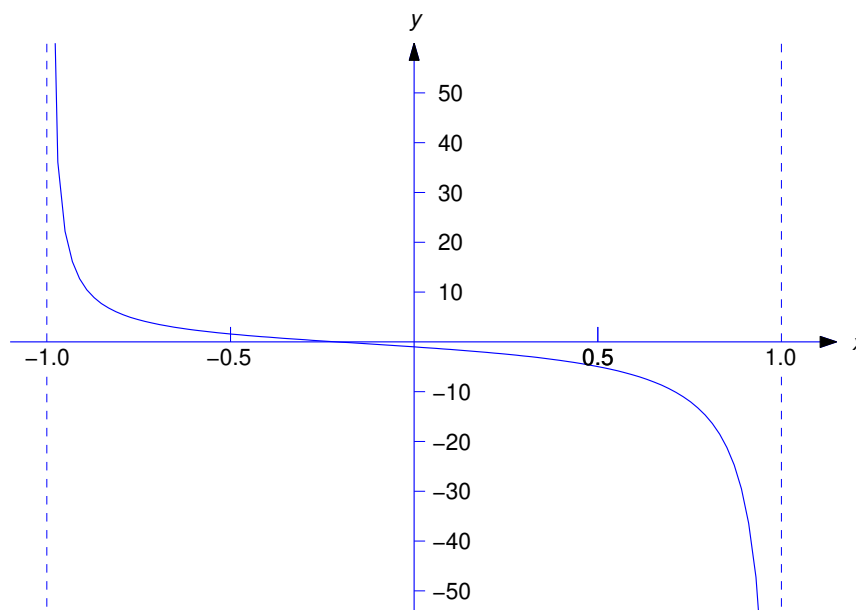


Figure 5.7.1 $y = \frac{2 \ln(x+1)}{x-1} + \frac{3x+1}{x^2-1}$

Section 6.1

Example 6.1.1 An object stretches a spring 6 inches in equilibrium.

- Set up the equation of motion and find its general solution.
- Find the displacement of the object for $t > 0$ if it's initially displaced 18 inches above equilibrium and given a downward velocity of 3 ft/s.

Solution (a) Setting $c = 0$ and $F = 0$ in (6.1.2) yields the equation of motion

$$my'' + ky = 0,$$

which we rewrite as

$$y'' + \frac{k}{m}y = 0. \quad (6.1.3)$$

Although we would need the weight of the object to obtain k from the equation $mg = k\Delta l$ we can obtain k/m from Δl alone; thus, $k/m = g/\Delta l$. Consistent with the units used in the problem statement, we take $g = 32 \text{ ft/s}^2$. Although Δl is stated in inches, we must convert it to feet to be consistent with this choice of g ; that is, $\Delta l = 1/2 \text{ ft}$. Therefore

$$\frac{k}{m} = \frac{32}{1/2} = 64$$

and (6.1.3) becomes

$$y'' + 64y = 0. \quad (6.1.4)$$

The characteristic equation of (6.1.4) is

$$r^2 + 64 = 0,$$

which has the zeros $r = \pm 8i$. Therefore the general solution of (6.1.4) is

$$y = c_1 \cos 8t + c_2 \sin 8t. \quad (6.1.5)$$

Solution (b) The initial upward displacement of 18 inches is positive and must be expressed in feet. The initial downward velocity is negative; thus,

$$y(0) = \frac{3}{2} \quad \text{and} \quad y'(0) = -3.$$

Differentiating (6.1.5) yields

$$y' = -8c_1 \sin 8t + 8c_2 \cos 8t. \quad (6.1.6)$$

Setting $t = 0$ in (6.1.5) and (6.1.6) and imposing the initial conditions shows that $c_1 = 3/2$ and $c_2 = -3/8$. Therefore

$$y = \frac{3}{2} \cos 8t - \frac{3}{8} \sin 8t,$$

where y is in feet (Figure 6.1.4).

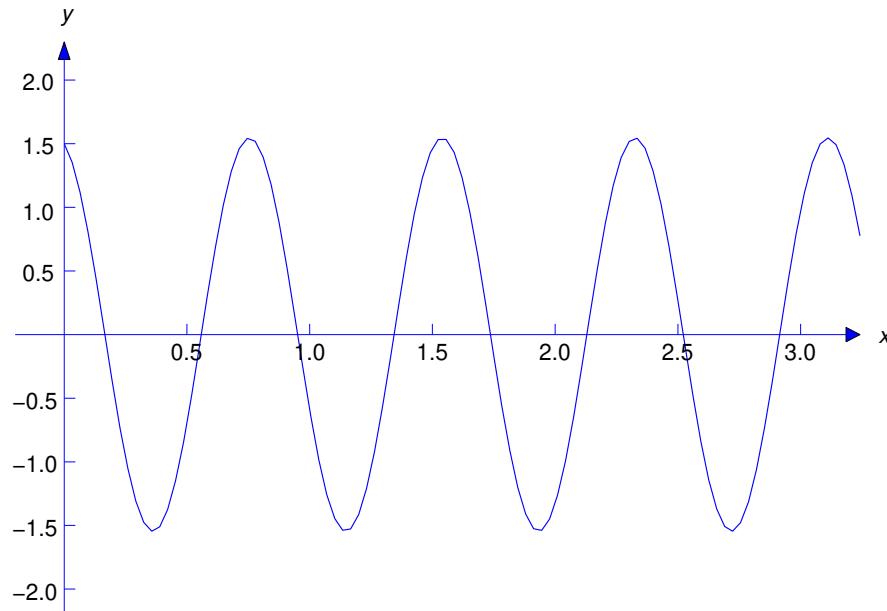


Figure 6.1.4 $y = \frac{3}{2} \cos 8t - \frac{3}{8} \sin 8t$

Example 6.1.2 We found the displacement of the object in Example 6.1.1 to be

$$y = \frac{3}{2} \cos 8t - \frac{3}{8} \sin 8t.$$

Find the frequency, period, amplitude, and phase angle of the motion.

Solution The frequency is $\omega_0 = 8$ rad/s, and the period is $T = 2\pi/\omega_0 = \pi/4$ s. Since $c_1 = 3/2$ and $c_2 = -3/8$, the amplitude is

$$R = \sqrt{c_1^2 + c_2^2} = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{3}{8}\right)^2} = \frac{3}{8}\sqrt{17}.$$

The phase angle is determined by

$$\cos \phi = \frac{\frac{3}{2}}{\frac{3}{8}\sqrt{17}} = \frac{4}{\sqrt{17}} \quad (6.1.11)$$

and

$$\sin \phi = \frac{-\frac{3}{8}}{\frac{3}{8}\sqrt{17}} = -\frac{1}{\sqrt{17}}. \quad (6.1.12)$$

Using a calculator, we see from (6.1.11) that

$$\phi \approx \pm .245 \text{ rad.}$$

Since $\sin \phi < 0$ (see (6.1.12)), the minus sign applies here; that is,

$$\phi \approx -.245 \text{ rad.}$$

Example 6.1.3 The natural length of a spring is 1 m. An object is attached to it and the length of the spring increases to 102 cm when the object is in equilibrium. Then the object is initially displaced downward 1 cm and given an upward velocity of 14 cm/s. Find the displacement for $t > 0$. Also, find the natural frequency, period, amplitude, and phase angle of the resulting motion. Express the answers in cgs units.

Solution In cgs units $g = 980 \text{ cm/s}^2$. Since $\Delta l = 2 \text{ cm}$, $\omega_0^2 = g/\Delta l = 490$. Therefore

$$y'' + 490y = 0, \quad y(0) = -1, \quad y'(0) = 14.$$

The general solution of the differential equation is

$$y = c_1 \cos 7\sqrt{10}t + c_2 \sin 7\sqrt{10}t,$$

so

$$y' = 7\sqrt{10} \left(-c_1 \sin 7\sqrt{10}t + c_2 \cos 7\sqrt{10}t \right).$$

Substituting the initial conditions into the last two equations yields $c_1 = -1$ and $c_2 = 2/\sqrt{10}$. Hence,

$$y = -\cos 7\sqrt{10}t + \frac{2}{\sqrt{10}} \sin 7\sqrt{10}t.$$

The frequency is $7\sqrt{10} \text{ rad/s}$, and the period is $T = 2\pi/(7\sqrt{10}) \text{ s}$. The amplitude is

$$R = \sqrt{c_1^2 + c_2^2} = \sqrt{(-1)^2 + \left(\frac{2}{\sqrt{10}}\right)^2} = \sqrt{\frac{7}{5}} \text{ cm.}$$

The phase angle is determined by

$$\cos \phi = \frac{c_1}{R} = -\sqrt{\frac{5}{7}} \quad \text{and} \quad \sin \phi = \frac{c_2}{R} = \sqrt{\frac{2}{7}}.$$

Therefore ϕ is in the second quadrant and

$$\phi = \cos^{-1} \left(-\sqrt{\frac{5}{7}} \right) \approx 2.58 \text{ rad.}$$

Example 6.1.4 Solve the initial value problem

$$y'' + \omega_0^2 y = \frac{F_0}{m} \cos \omega t, \quad y(0) = 0, \quad y'(0) = 0, \quad (6.1.14)$$

given that $\omega \neq \omega_0$.

Solution We first obtain a particular solution of (6.1.13) by the method of undetermined coefficients. Since $\omega \neq \omega_0$, $\cos \omega t$ isn't a solution of the complementary equation

$$y'' + \omega_0^2 y = 0.$$

Therefore (6.1.13) has a particular solution of the form

$$y_p = A \cos \omega t + B \sin \omega t.$$

Since

$$y_p'' = -\omega^2 (A \cos \omega t + B \sin \omega t),$$

$$y_p'' + \omega_0^2 y_p = \frac{F_0}{m} \cos \omega t$$

if and only if

$$(\omega_0^2 - \omega^2) (A \cos \omega t + B \sin \omega t) = \frac{F_0}{m} \cos \omega t.$$

This holds if and only if

$$A = \frac{F_0}{m(\omega_0^2 - \omega^2)} \quad \text{and} \quad B = 0,$$

so

$$y_p = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t.$$

The general solution of (6.1.13) is

$$y = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos \omega t + c_1 \cos \omega_0 t + c_2 \sin \omega_0 t, \quad (6.1.15)$$

so

$$y' = \frac{-\omega F_0}{m(\omega_0^2 - \omega^2)} \sin \omega t + \omega_0 (-c_1 \sin \omega_0 t + c_2 \cos \omega_0 t).$$

The initial conditions $y(0) = 0$ and $y'(0) = 0$ in (6.1.14) imply that

$$c_1 = -\frac{F_0}{m(\omega_0^2 - \omega^2)} \quad \text{and} \quad c_2 = 0.$$

Substituting these into (6.1.15) yields

$$y = \frac{F_0}{m(\omega_0^2 - \omega^2)} (\cos \omega t - \cos \omega_0 t). \quad (6.1.16)$$

It is revealing to write this in a different form. We start with the trigonometric identities

$$\begin{aligned} \cos(\alpha - \beta) &= \cos \alpha \cos \beta + \sin \alpha \sin \beta \\ \cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta. \end{aligned}$$

Subtracting the second identity from the first yields

$$\cos(\alpha - \beta) - \cos(\alpha + \beta) = 2 \sin \alpha \sin \beta \quad (6.1.17)$$

Now let

$$\alpha - \beta = \omega t \quad \text{and} \quad \alpha + \beta = \omega_0 t, \quad (6.1.18)$$

so that

$$\alpha = \frac{(\omega_0 + \omega)t}{2} \quad \text{and} \quad \beta = \frac{(\omega_0 - \omega)t}{2}. \quad (6.1.19)$$

Substituting (6.1.18) and (6.1.19) into (6.1.17) yields

$$\cos \omega t - \cos \omega_0 t = 2 \sin \frac{(\omega_0 - \omega)t}{2} \sin \frac{(\omega_0 + \omega)t}{2},$$

and substituting this into (6.1.16) yields

$$y = R(t) \sin \frac{(\omega_0 + \omega)t}{2}, \quad (6.1.20)$$

where

$$R(t) = \frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin \frac{(\omega_0 - \omega)t}{2}. \quad (6.1.21)$$

From (6.1.20) we can regard y as a sinusoidal variation with frequency $(\omega_0 + \omega)/2$ and variable amplitude $|R(t)|$. In Figure 6.1.6 the dashed curve above the t axis is $y = |R(t)|$, the dashed curve below the t axis is $y = -|R(t)|$, and the displacement y appears as an oscillation bounded by them. The oscillation of y for t on an interval between successive zeros of $R(t)$ is called a *beat*.

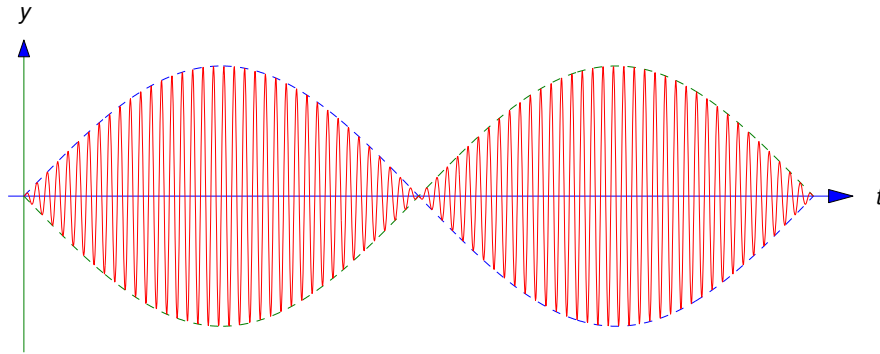


Figure 6.1.6 Undamped oscillation with beats

You can see from (6.1.20) and (6.1.21) that

$$|y(t)| \leq \frac{2|F_0|}{m|\omega_0^2 - \omega^2|};$$

moreover, if $\omega + \omega_0$ is sufficiently large compared with $\omega - \omega_0$, then $|y|$ assumes values close to (perhaps equal to) this upper bound during each beat. However, the oscillation remains bounded for all t . (This assumes that the spring can withstand deflections of this size and continue to obey Hooke's law.) The next example shows that this isn't so if $\omega = \omega_0$.

Example 6.1.5 Find the general solution of

$$y'' + \omega_0^2 y = \frac{F_0}{m} \cos \omega_0 t. \quad (6.1.22)$$

Solution We first obtain a particular solution y_p of (6.1.22). Since $\cos \omega_0 t$ is a solution of the complementary equation, the form for y_p is

$$y_p = t(A \cos \omega_0 t + B \sin \omega_0 t). \quad (6.1.23)$$

Then

$$y'_p = A \cos \omega_0 t + B \sin \omega_0 t + \omega_0 t(-A \sin \omega_0 t + B \cos \omega_0 t)$$

and

$$y''_p = 2\omega_0(-A \sin \omega_0 t + B \cos \omega_0 t) - \omega_0^2 t(A \cos \omega_0 t + B \sin \omega_0 t). \quad (6.1.24)$$

From (6.1.23) and (6.1.24), we see that y_p satisfies (6.1.22) if

$$-2A\omega_0 \sin \omega_0 t + 2B\omega_0 \cos \omega_0 t = \frac{F_0}{m} \cos \omega_0 t;$$

that is, if

$$A = 0 \quad \text{and} \quad B = \frac{F_0}{2m\omega_0}.$$

Therefore

$$y_p = \frac{F_0 t}{2m\omega_0} \sin \omega_0 t$$

is a particular solution of (6.1.22). The general solution of (6.1.22) is

$$y = \frac{F_0 t}{2m\omega_0} \sin \omega_0 t + c_1 \cos \omega_0 t + c_2 \sin \omega_0 t.$$

The graph of y_p is shown in Figure 6.1.7, where it can be seen that y_p oscillates between the dashed lines

$$y = \frac{F_0 t}{2m\omega_0} \quad \text{and} \quad y = -\frac{F_0 t}{2m\omega_0}$$

with increasing amplitude that approaches ∞ as $t \rightarrow \infty$. Of course, this means that the spring must eventually fail to obey Hooke's law or break. ■

Section 6.2

Example 6.2.1 Suppose a 64 lb weight stretches a spring 6 inches in equilibrium and a dashpot provides a damping force of c lb for each ft/sec of velocity.

- (a) Write the equation of motion of the object and determine the value of c for which the motion is critically damped.

- (b) Find the displacement y for $t > 0$ if the motion is critically damped and the initial conditions are $y(0) = 1$ and $y'(0) = 20$.
- (c) Find the displacement y for $t > 0$ if the motion is critically damped and the initial conditions are $y(0) = 1$ and $y'(0) = -20$.

Solution (a) Here $m = 2$ slugs and $k = 64/.5 = 128$ lb/ft. Therefore the equation of motion (6.2.1) is

$$2y'' + cy' + 128y = 0. \quad (6.2.4)$$

The characteristic equation is

$$2r^2 + cr + 128 = 0,$$

which has roots

$$r = \frac{-c \pm \sqrt{c^2 - 8 \cdot 128}}{4}.$$

Therefore the damping is critical if

$$c = \sqrt{8 \cdot 128} = 32 \text{ lb-sec/ft.}$$

Solution (b) Setting $c = 32$ in (6.2.4) and cancelling the common factor 2 yields

$$y'' + 16y' + 64y = 0.$$

The characteristic equation is

$$r^2 + 16r + 64 = (r + 8)^2 = 0.$$

Hence, the general solution is

$$y = e^{-8t}(c_1 + c_2 t). \quad (6.2.5)$$

Differentiating this yields

$$y' = -8y + c_2 e^{-8t}. \quad (6.2.6)$$

Imposing the initial conditions $y(0) = 1$ and $y'(0) = 20$ in the last two equations shows that $1 = c_1$ and $20 = -8 + c_2$. Hence, the solution of the initial value problem is

$$y = e^{-8t}(1 + 28t).$$

Therefore the object approaches equilibrium from above as $t \rightarrow \infty$. There's no oscillation.

Solution (c) Imposing the initial conditions $y(0) = 1$ and $y'(0) = -20$ in (6.2.5) and (6.2.6) yields $1 = c_1$ and $-20 = -8 + c_2$. Hence, the solution of this initial value problem is

$$y = e^{-8t}(1 - 12t).$$

Therefore the object moves downward through equilibrium just once, and then approaches equilibrium from below as $t \rightarrow \infty$. Again, there's no oscillation. The solutions of these two initial value problems are graphed in Figure 6.2.2.

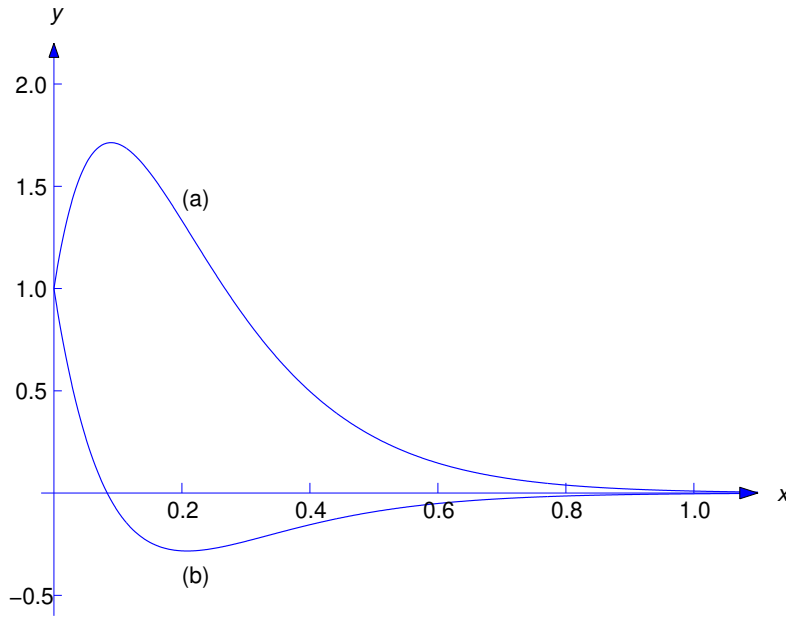


Figure 6.2.2 (a) $y = e^{-8t}(1 + 28t)$ (b) $y = e^{-8t}(1 - 12t)$

Example 6.2.2 Find the displacement of the object in Example 6.2.1 if the damping constant is $c = 4$ lb–sec/ft and the initial conditions are $y(0) = 1.5$ ft and $y'(0) = -3$ ft/sec.

Solution With $c = 4$, the equation of motion (6.2.4) becomes

$$y'' + 2y' + 64y = 0 \quad (6.2.7)$$

after cancelling the common factor 2. The characteristic equation

$$r^2 + 2r + 64 = 0$$

has complex conjugate roots

$$r = \frac{-2 \pm \sqrt{4 - 4 \cdot 64}}{2} = -1 \pm 3\sqrt{7}i.$$

Therefore the motion is underdamped and the general solution of (6.2.7) is

$$y = e^{-t}(c_1 \cos 3\sqrt{7}t + c_2 \sin 3\sqrt{7}t).$$

Differentiating this yields

$$y' = -y + 3\sqrt{7}e^{-t}(-c_1 \sin 3\sqrt{7}t + c_2 \cos 3\sqrt{7}t).$$

Imposing the initial conditions $y(0) = 1.5$ and $y'(0) = -3$ in the last two equations yields $1.5 = c_1$ and $-3 = -1.5 + 3\sqrt{7}c_2$. Hence, the solution of the initial value problem is

$$y = e^{-t}\left(\frac{3}{2} \cos 3\sqrt{7}t - \frac{1}{2\sqrt{7}} \sin 3\sqrt{7}t\right). \quad (6.2.8)$$

The amplitude of the function in parentheses is

$$R = \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{1}{2\sqrt{7}}\right)^2} = \sqrt{\frac{9}{4} + \frac{1}{4 \cdot 7}} = \sqrt{\frac{64}{4 \cdot 7}} = \frac{4}{\sqrt{7}}.$$

Therefore we can rewrite (6.2.8) as

$$y = \frac{4}{\sqrt{7}} e^{-t} \cos(3\sqrt{7}t - \phi),$$

where

$$\cos \phi = \frac{3}{2R} = \frac{3\sqrt{7}}{8} \quad \text{and} \quad \sin \phi = -\frac{1}{2\sqrt{7}R} = -\frac{1}{8}.$$

Therefore $\phi \cong -.125$ radians.

Example 6.2.3 Let the damping constant in Example 1 be $c = 40$ lb-sec/ft. Find the displacement y for $t > 0$ if $y(0) = 1$ and $y'(0) = 1$.

Solution With $c = 40$, the equation of motion (6.2.4) reduces to

$$y'' + 20y' + 64y = 0 \tag{6.2.9}$$

after cancelling the common factor 2. The characteristic equation

$$r^2 + 20r + 64 = (r + 16)(r + 4) = 0$$

has the roots $r_1 = -4$ and $r_2 = -16$. Therefore the general solution of (6.2.9) is

$$y = c_1 e^{-4t} + c_2 e^{-16t}. \tag{6.2.10}$$

Differentiating this yields

$$y' = -4c_1 e^{-4t} - 16c_2 e^{-16t}.$$

The last two equations and the initial conditions $y(0) = 1$ and $y'(0) = 1$ imply that

$$\begin{array}{rcl} c_1 & + & c_2 = 1 \\ -4c_1 & - & 16c_2 = 1. \end{array}$$

The solution of this system is $c_1 = 17/12$, $c_2 = -5/12$. Substituting these into (6.2.10) yields

$$y = \frac{17}{12} e^{-4t} - \frac{5}{12} e^{-16t}$$

as the solution of the given initial value problem (Figure 6.2.3).

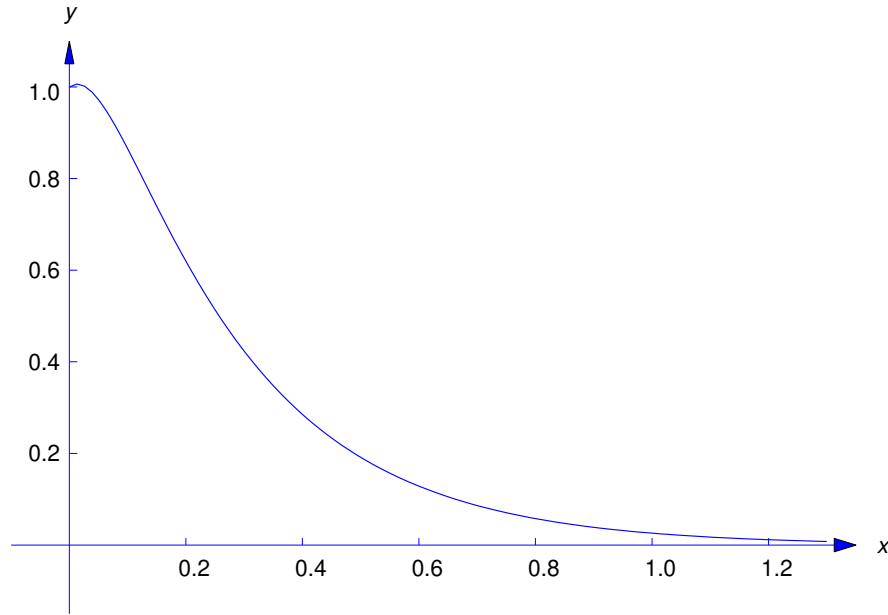


Figure 6.2.3 $y = \frac{17}{12}e^{-4t} - \frac{5}{12}e^{-16t}$

Section 6.3

Example 6.3.1 At $t = 0$ a current of 2 amperes flows in an RLC circuit with resistance $R = 40$ ohms, inductance $L = .2$ henrys, and capacitance $C = 10^{-5}$ farads. Find the current flowing in the circuit at $t > 0$ if the initial charge on the capacitor is 1 coulomb. Assume that $E(t) = 0$ for $t > 0$.

Solution The equation for the charge Q is

$$\frac{1}{5}Q'' + 40Q' + 10000Q = 0,$$

or

$$Q'' + 200Q' + 50000Q = 0. \quad (6.3.13)$$

Therefore we must solve the initial value problem

$$Q'' + 200Q' + 50000Q = 0, \quad Q(0) = 1, \quad Q'(0) = 2. \quad (6.3.14)$$

The desired current is the derivative of the solution of this initial value problem.

The characteristic equation of (6.3.13) is

$$r^2 + 200r + 50000 = 0,$$

which has complex zeros $r = -100 \pm 200i$. Therefore the general solution of (6.3.13) is

$$Q = e^{-100t}(c_1 \cos 200t + c_2 \sin 200t). \quad (6.3.15)$$

Differentiating this and collecting like terms yields

$$Q' = -e^{-100t}[(100c_1 - 200c_2) \cos 200t + (100c_2 + 200c_1) \sin 200t]. \quad (6.3.16)$$

To find the solution of the initial value problem (6.3.14), we set $t = 0$ in (6.3.15) and (6.3.16) to obtain

$$c_1 = Q(0) = 1 \quad \text{and} \quad -100c_1 + 200c_2 = Q'(0) = 2;$$

therefore, $c_1 = 1$ and $c_2 = 51/100$, so

$$Q = e^{-100t} \left(\cos 200t + \frac{51}{100} \sin 200t \right)$$

is the solution of (6.3.14). Differentiating this yields

$$I = e^{-100t} (2 \cos 200t - 251 \sin 200t).$$

Example 6.3.2 Find the amplitude-phase form of the steady state current in the RLC circuit in Figure 6.3.1 if the impressed voltage, provided by an alternating current generator, is $E(t) = E_0 \cos \omega t$.

Solution We'll first find the steady state charge on the capacitor as a particular solution of

$$LQ'' + RQ' + \frac{1}{C}Q = E_0 \cos \omega t.$$

To do this, we'll simply reinterpret a result obtained in Section 6.2, where we found that the steady state solution of

$$my'' + cy' + ky = F_0 \cos \omega t$$

is

$$y_p = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \cos(\omega t - \phi),$$

where

$$\cos \phi = \frac{k - m\omega^2}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \quad \text{and} \quad \sin \phi = \frac{c\omega}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}}.$$

(See Equations (6.2.14) and (6.2.15).) By making the appropriate changes in the symbols (according to Table 6.3.2) we obtain the steady state charge

$$Q_p = \frac{E_0}{\sqrt{(1/C - L\omega^2)^2 + R^2\omega^2}} \cos(\omega t - \phi),$$

where

$$\cos \phi = \frac{1/C - L\omega^2}{\sqrt{(1/C - L\omega^2)^2 + R^2\omega^2}} \quad \text{and} \quad \sin \phi = \frac{R\omega}{\sqrt{(1/C - L\omega^2)^2 + R^2\omega^2}}.$$

Therefore the steady state current in the circuit is

$$I_p = Q'_p = -\frac{\omega E_0}{\sqrt{(1/C - L\omega^2)^2 + R^2\omega^2}} \sin(\omega t - \phi).$$

Section 7.1

Example 7.1.1 Find the radius of convergence of the series:

$$(a) \sum_{n=0}^{\infty} n!x^n \quad (b) \sum_{n=10}^{\infty} (-1)^n \frac{x^n}{n!} \quad (c) \sum_{n=0}^{\infty} 2^n n^2 (x-1)^n.$$

Solution (a) Here $a_n = n!$, so

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{(n+1)!}{n!} = \lim_{n \rightarrow \infty} (n+1) = \infty.$$

Hence, $R = 0$.

Solution (b) Here $a_n = (-1)^n/n!$ for $n \geq N = 10$, so

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{n!}{(n+1)!} = \lim_{n \rightarrow \infty} \frac{1}{n+1} = 0.$$

Hence, $R = \infty$.

Solution (c) Here $a_n = 2^n n^2$, so

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \frac{2^{n+1}(n+1)^2}{2^n n^2} = 2 \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^2 = 2.$$

Hence, $R = 1/2$.

Example 7.1.2 Let $f(x) = \sin x$. From (7.1.3),

$$f(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

From (7.1.6),

$$f'(x) = \sum_{n=0}^{\infty} (-1)^n \frac{d}{dx} \left[\frac{x^{2n+1}}{(2n+1)!} \right] = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!},$$

which is the series (7.1.4) for $\cos x$.

Example 7.1.3 Rewrite the following power series from (7.1.7) and (7.1.8) so that the general term in each is a constant multiple of $(x - x_0)^n$:

$$(a) \sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2} \quad (b) \sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(x-x_0)^{n-k}.$$

Solution (a) Replacing n by $n+2$ in the general term and subtracting 2 from the lower limit of summation yields

$$\sum_{n=2}^{\infty} n(n-1)a_n(x-x_0)^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x-x_0)^n.$$

Solution (b) Replacing n by $n+k$ in the general term and subtracting k from the lower limit of summation yields

$$\sum_{n=k}^{\infty} n(n-1)\cdots(n-k+1)a_n(x-x_0)^{n-k} = \sum_{n=0}^{\infty} (n+k)(n+k-1)\cdots(n+1)a_{n+k}(x-x_0)^n.$$

Example 7.1.4 Given that

$$f(x) = \sum_{n=0}^{\infty} a_n x^n,$$

write the function xf'' as a power series in which the general term is a constant multiple of x^n .

Solution From Theorem 7.1.4 with $x_0 = 0$,

$$f''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Therefore

$$xf''(x) = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1}.$$

Replacing n by $n+1$ in the general term and subtracting 1 from the lower limit of summation yields

$$xf''(x) = \sum_{n=1}^{\infty} (n+1)na_{n+1}x^n.$$

We can also write this as

$$xf''(x) = \sum_{n=0}^{\infty} (n+1)na_{n+1}x^n,$$

since the first term in this last series is zero. (We'll see later that sometimes it's useful to include zero terms at the beginning of a series.)

Example 7.1.5 Find the Maclaurin series for $\cosh x$ as a linear combination of the Maclaurin series for e^x and e^{-x} .

Solution By definition,

$$\cosh x = \frac{1}{2}e^x + \frac{1}{2}e^{-x}.$$

Since

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \text{and} \quad e^{-x} = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!},$$

it follows that

$$\cosh x = \sum_{n=0}^{\infty} \frac{1}{2} [1 + (-1)^n] \frac{x^n}{n!}. \quad (7.1.12)$$

Since

$$\frac{1}{2} [1 + (-1)^n] = \begin{cases} 1 & \text{if } n = 2m, \text{ an even integer,} \\ 0 & \text{if } n = 2m + 1, \text{ an odd integer,} \end{cases}$$

we can rewrite (7.1.12) more simply as

$$\cosh x = \sum_{m=0}^{\infty} \frac{x^{2m}}{(2m)!}.$$

This result is valid on $(-\infty, \infty)$, since this is the open interval of convergence of the Maclaurin series for e^x and e^{-x} .

Example 7.1.6 Suppose

$$y = \sum_{n=0}^{\infty} a_n x^n$$

on an open interval I that contains the origin.

(a) Express

$$(2 - x)y'' + 2y$$

as a power series in x on I .

(b) Use the result of (a) to find necessary and sufficient conditions on the coefficients $\{a_n\}$ for y to be a solution of the homogeneous equation

$$(2 - x)y'' + 2y = 0 \quad (7.1.13)$$

on I .

Solution (a) From (7.1.7) with $x_0 = 0$,

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}.$$

Therefore

$$\begin{aligned} (2 - x)y'' + 2y &= 2y'' - xy'' + 2y \\ &= \sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} + \sum_{n=0}^{\infty} 2a_n x^n. \end{aligned} \quad (7.1.14)$$

To combine the three series we shift indices in the first two to make their general terms constant multiples of x^n ; thus,

$$\sum_{n=2}^{\infty} 2n(n-1)a_n x^{n-2} = \sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2} x^n \quad (7.1.15)$$

and

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-1} = \sum_{n=1}^{\infty} (n+1)na_{n+1} x^n = \sum_{n=0}^{\infty} (n+1)na_{n+1} x^n, \quad (7.1.16)$$

where we added a zero term in the last series so that when we substitute from (7.1.15) and (7.1.16) into (7.1.14) all three series will start with $n = 0$; thus,

$$(2 - x)y'' + 2y = \sum_{n=0}^{\infty} [2(n+2)(n+1)a_{n+2} - (n+1)na_{n+1} + 2a_n] x^n. \quad (7.1.17)$$

Solution (b) From (7.1.17) we see that y satisfies (7.1.13) on I if

$$2(n+2)(n+1)a_{n+2} - (n+1)na_{n+1} + 2a_n = 0, \quad n = 0, 1, 2, \dots \quad (7.1.18)$$

Conversely, Theorem 7.1.6 (b) implies that if $y = \sum_{n=0}^{\infty} a_n x^n$ satisfies (7.1.13) on I , then (7.1.18) holds.

Example 7.1.7 Suppose

$$y = \sum_{n=0}^{\infty} a_n(x-1)^n$$

on an open interval I that contains $x_0 = 1$. Express the function

$$(1+x)y'' + 2(x-1)^2y' + 3y \quad (7.1.19)$$

as a power series in $x-1$ on I .

Solution Since we want a power series in $x-1$, we rewrite the coefficient of y'' in (7.1.19) as $1+x = 2+(x-1)$, so (7.1.19) becomes

$$2y'' + (x-1)y'' + 2(x-1)^2y' + 3y.$$

From (7.1.6) and (7.1.7) with $x_0 = 1$,

$$y' = \sum_{n=1}^{\infty} na_n(x-1)^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-2}.$$

Therefore

$$\begin{aligned} 2y'' &= \sum_{n=2}^{\infty} 2n(n-1)a_n(x-1)^{n-2}, \\ (x-1)y'' &= \sum_{n=2}^{\infty} n(n-1)a_n(x-1)^{n-1}, \\ 2(x-1)^2y' &= \sum_{n=1}^{\infty} 2na_n(x-1)^{n+1}, \\ 3y &= \sum_{n=0}^{\infty} 3a_n(x-1)^n. \end{aligned}$$

Before adding these four series we shift indices in the first three so that their general terms become constant multiples of $(x-1)^n$. This yields

$$2y'' = \sum_{n=0}^{\infty} 2(n+2)(n+1)a_{n+2}(x-1)^n, \quad (7.1.20)$$

$$(x-1)y'' = \sum_{n=0}^{\infty} (n+1)na_{n+1}(x-1)^n, \quad (7.1.21)$$

$$2(x-1)^2y' = \sum_{n=1}^{\infty} 2(n-1)a_{n-1}(x-1)^n, \quad (7.1.22)$$

$$3y = \sum_{n=0}^{\infty} 3a_n(x-1)^n, \quad (7.1.23)$$

where we added initial zero terms to the series in (7.1.21) and (7.1.22). Adding (7.1.20)–(7.1.23) yields

$$\begin{aligned} (1+x)y'' + 2(x-1)^2y' + 3y &= 2y'' + (x-1)y'' + 2(x-1)^2y' + 3y \\ &= \sum_{n=0}^{\infty} b_n(x-1)^n, \end{aligned}$$

where

$$b_0 = 4a_2 + 3a_0, \quad (7.1.24)$$

$$b_n = 2(n+2)(n+1)a_{n+2} + (n+1)na_{n+1} + 2(n-1)a_{n-1} + 3a_n, \quad n \geq 1. \quad (7.1.25)$$

The formula (7.1.24) for b_0 can't be obtained by setting $n = 0$ in (7.1.25), since the summation in (7.1.22) begins with $n = 1$, while those in (7.1.20), (7.1.21), and (7.1.23) begin with $n = 0$.

Section 7.2

Example 7.2.1 Let x_0 be an arbitrary real number. Find the power series in $x - x_0$ for the general solution of

$$y'' + y = 0. \quad (7.2.9)$$

Solution Here

$$Ly = y'' + y.$$

If

$$y = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

then

$$y'' = \sum_{n=2}^{\infty} n(n-1)a_n (x - x_0)^{n-2},$$

so

$$Ly = \sum_{n=2}^{\infty} n(n-1)a_n (x - x_0)^{n-2} + \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

To collect coefficients of like powers of $x - x_0$, we shift the summation index in the first sum. This yields

$$Ly = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} (x - x_0)^n + \sum_{n=0}^{\infty} a_n (x - x_0)^n = \sum_{n=0}^{\infty} b_n (x - x_0)^n,$$

with

$$b_n = (n+2)(n+1)a_{n+2} + a_n.$$

Therefore $Ly = 0$ if and only if

$$a_{n+2} = \frac{-a_n}{(n+2)(n+1)}, \quad n \geq 0, \quad (7.2.10)$$

where a_0 and a_1 are arbitrary. Since the indices on the left and right sides of (7.2.10) differ by two, we write (7.2.10) separately for n even ($n = 2m$) and n odd ($n = 2m + 1$). This yields

$$a_{2m+2} = \frac{-a_{2m}}{(2m+2)(2m+1)}, \quad m \geq 0, \quad (7.2.11)$$

and

$$a_{2m+3} = \frac{-a_{2m+1}}{(2m+3)(2m+2)}, \quad m \geq 0. \quad (7.2.12)$$

Computing the coefficients of the even powers of $x - x_0$ from (7.2.11) yields

$$\begin{aligned} a_2 &= -\frac{a_0}{2 \cdot 1} \\ a_4 &= -\frac{a_2}{4 \cdot 3} = -\frac{1}{4 \cdot 3} \left(-\frac{a_0}{2 \cdot 1} \right) = \frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1}, \\ a_6 &= -\frac{a_4}{6 \cdot 5} = -\frac{1}{6 \cdot 5} \left(\frac{a_0}{4 \cdot 3 \cdot 2 \cdot 1} \right) = -\frac{a_0}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}, \end{aligned}$$

and, in general,

$$a_{2m} = (-1)^m \frac{a_0}{(2m)!}, \quad m \geq 0. \quad (7.2.13)$$

Computing the coefficients of the odd powers of $x - x_0$ from (7.2.12) yields

$$\begin{aligned} a_3 &= -\frac{a_1}{3 \cdot 2} \\ a_5 &= -\frac{a_3}{5 \cdot 4} = -\frac{1}{5 \cdot 4} \left(-\frac{a_1}{3 \cdot 2} \right) = \frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2}, \\ a_7 &= -\frac{a_5}{7 \cdot 6} = -\frac{1}{7 \cdot 6} \left(\frac{a_1}{5 \cdot 4 \cdot 3 \cdot 2} \right) = -\frac{a_1}{7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}, \end{aligned}$$

and, in general,

$$a_{2m+1} = \frac{(-1)^m a_1}{(2m+1)!} \quad m \geq 0. \quad (7.2.14)$$

Thus, the general solution of (7.2.9) can be written as

$$y = \sum_{m=0}^{\infty} a_{2m} (x - x_0)^{2m} + \sum_{m=0}^{\infty} a_{2m+1} (x - x_0)^{2m+1},$$

or, from (7.2.13) and (7.2.14), as

$$y = a_0 \sum_{m=0}^{\infty} (-1)^m \frac{(x - x_0)^{2m}}{(2m)!} + a_1 \sum_{m=0}^{\infty} (-1)^m \frac{(x - x_0)^{2m+1}}{(2m+1)!}. \quad (7.2.15)$$

If we recall from calculus that

$$\sum_{m=0}^{\infty} (-1)^m \frac{(x - x_0)^{2m}}{(2m)!} = \cos(x - x_0) \quad \text{and} \quad \sum_{m=0}^{\infty} (-1)^m \frac{(x - x_0)^{2m+1}}{(2m+1)!} = \sin(x - x_0),$$

then (7.2.15) becomes

$$y = a_0 \cos(x - x_0) + a_1 \sin(x - x_0),$$

which should look familiar. ■

Example 7.2.2 Find the power series in x for the general solution of

$$(1 + 2x^2)y'' + 6xy' + 2y = 0. \quad (7.2.17)$$

Solution Here

$$Ly = (1 + 2x^2)y'' + 6xy' + 2y.$$

If

$$y = \sum_{n=0}^{\infty} a_n x^n$$

then

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2},$$

so

$$\begin{aligned} Ly &= (1 + 2x^2) \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 6x \sum_{n=1}^{\infty} n a_n x^{n-1} + 2 \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=0}^{\infty} [2n(n-1) + 6n + 2] a_n x^n \\ &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 2 \sum_{n=0}^{\infty} (n+1)^2 a_n x^n. \end{aligned}$$

To collect coefficients of x^n , we shift the summation index in the first sum. This yields

$$Ly = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + 2 \sum_{n=0}^{\infty} (n+1)^2 a_n x^n = \sum_{n=0}^{\infty} b_n x^n,$$

with

$$b_n = (n+2)(n+1) a_{n+2} + 2(n+1)^2 a_n, \quad n \geq 0.$$

To obtain solutions of (7.2.17), we set $b_n = 0$ for $n \geq 0$. This is equivalent to the recurrence relation

$$a_{n+2} = -2 \frac{n+1}{n+2} a_n, \quad n \geq 0. \quad (7.2.18)$$

Since the indices on the left and right differ by two, we write (7.2.18) separately for $n = 2m$ and $n = 2m + 1$, as in Example 7.2.1. This yields

$$a_{2m+2} = -2 \frac{2m+1}{2m+2} a_{2m} = -\frac{2m+1}{m+1} a_{2m}, \quad m \geq 0, \quad (7.2.19)$$

and

$$a_{2m+3} = -2 \frac{2m+2}{2m+3} a_{2m+1} = -4 \frac{m+1}{2m+3} a_{2m+1}, \quad m \geq 0. \quad (7.2.20)$$

Computing the coefficients of even powers of x from (7.2.19) yields

$$\begin{aligned} a_2 &= -\frac{1}{1} a_0, \\ a_4 &= -\frac{3}{2} a_2 = \left(-\frac{3}{2}\right) \left(-\frac{1}{1}\right) a_0 = \frac{1 \cdot 3}{1 \cdot 2} a_0, \\ a_6 &= -\frac{5}{3} a_4 = -\frac{5}{3} \left(\frac{1 \cdot 3}{1 \cdot 2}\right) a_0 = -\frac{1 \cdot 3 \cdot 5}{1 \cdot 2 \cdot 3} a_0, \\ a_8 &= -\frac{7}{4} a_6 = -\frac{7}{4} \left(-\frac{1 \cdot 3 \cdot 5}{1 \cdot 2 \cdot 3}\right) a_0 = \frac{1 \cdot 3 \cdot 5 \cdot 7}{1 \cdot 2 \cdot 3 \cdot 4} a_0. \end{aligned}$$

In general,

$$a_{2m} = (-1)^m \frac{\prod_{j=1}^m (2j-1)}{m!} a_0, \quad m \geq 0. \quad (7.2.21)$$

(Note that (7.2.21) is correct for $m = 0$ because we defined $\prod_{j=1}^0 b_j = 1$ for any b_j .)

Computing the coefficients of odd powers of x from (7.2.20) yields

$$\begin{aligned} a_3 &= -4 \frac{1}{3} a_1, \\ a_5 &= -4 \frac{2}{5} a_3 = -4 \frac{2}{5} \left(-4 \frac{1}{3} \right) a_1 = 4^2 \frac{1 \cdot 2}{3 \cdot 5} a_1, \\ a_7 &= -4 \frac{3}{7} a_5 = -4 \frac{3}{7} \left(4^2 \frac{1 \cdot 2}{3 \cdot 5} \right) a_1 = -4^3 \frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7} a_1, \\ a_9 &= -4 \frac{4}{9} a_7 = -4 \frac{4}{9} \left(4^3 \frac{1 \cdot 2 \cdot 3}{3 \cdot 5 \cdot 7} \right) a_1 = 4^4 \frac{1 \cdot 2 \cdot 3 \cdot 4}{3 \cdot 5 \cdot 7 \cdot 9} a_1. \end{aligned}$$

In general,

$$a_{2m+1} = \frac{(-1)^m 4^m m!}{\prod_{j=1}^m (2j+1)} a_1, \quad m \geq 0. \quad (7.2.22)$$

From (7.2.21) and (7.2.22),

$$y = a_0 \sum_{m=0}^{\infty} (-1)^m \frac{\prod_{j=1}^m (2j-1)}{m!} x^{2m} + a_1 \sum_{m=0}^{\infty} (-1)^m \frac{4^m m!}{\prod_{j=1}^m (2j+1)} x^{2m+1}.$$

is the power series in x for the general solution of (7.2.17). Since $P_0(x) = 1 + 2x^2$ has no real zeros, Theorem 5.1.1 implies that every solution of (7.2.17) is defined on $(-\infty, \infty)$. However, since $P_0(\pm i/\sqrt{2}) = 0$, Theorem 7.2.1 implies only that the power series converges in $(-1/\sqrt{2}, 1/\sqrt{2})$ for any choice of a_0 and a_1 .

Example 7.2.3 Find the power series in $x - 1$ for the general solution of

$$(2 + 4x - 2x^2)y'' - 12(x-1)y' - 12y = 0. \quad (7.2.28)$$

Solution We must first write the coefficient $P_0(x) = 2 + 4x - 2x^2$ in powers of $x - 1$. To do this, we write $x = (x - 1) + 1$ in $P_0(x)$ and then expand the terms, collecting powers of $x - 1$; thus,

$$\begin{aligned} 2 + 4x - 2x^2 &= 2 + 4[(x-1) + 1] - 2[(x-1) + 1]^2 \\ &= 4 - 2(x-1)^2. \end{aligned}$$

Therefore we can rewrite (7.2.28) as

$$(4 - 2(x-1)^2)y'' - 12(x-1)y' - 12y = 0,$$

or, equivalently,

$$\left(1 - \frac{1}{2}(x-1)^2\right)y'' - 3(x-1)y' - 3y = 0.$$

This is of the form (7.2.23) with $\alpha = -1/2$, $\beta = -3$, and $\gamma = -3$. Therefore, from (7.2.25)

$$p(n) = -\frac{n(n-1)}{2} - 3n - 3 = -\frac{(n+2)(n+3)}{2}.$$

Hence, Theorem 7.2.2 implies that

$$\begin{aligned} a_{2m+2} &= -\frac{p(2m)}{(2m+2)(2m+1)}a_{2m} \\ &= \frac{(2m+2)(2m+3)}{2(2m+2)(2m+1)}a_{2m} = \frac{2m+3}{2(2m+1)}a_{2m}, \quad m \geq 0 \end{aligned}$$

and

$$\begin{aligned} a_{2m+3} &= -\frac{p(2m+1)}{(2m+3)(2m+2)}a_{2m+1} \\ &= \frac{(2m+3)(2m+4)}{2(2m+3)(2m+2)}a_{2m+1} = \frac{m+2}{2(m+1)}a_{2m+1}, \quad m \geq 0. \end{aligned}$$

We leave it to you to show that

$$a_{2m} = \frac{2m+1}{2^m}a_0 \quad \text{and} \quad a_{2m+1} = \frac{m+1}{2^m}a_1, \quad m \geq 0,$$

which implies that the power series in $x-1$ for the general solution of (7.2.28) is

$$y = a_0 \sum_{m=0}^{\infty} \frac{2m+1}{2^m} (x-1)^{2m} + a_1 \sum_{m=0}^{\infty} \frac{m+1}{2^m} (x-1)^{2m+1}. \quad \blacksquare$$

Example 7.2.4 Compute $a_0, a_1, \dots, a_7$ in the series solution $y = \sum_{n=0}^{\infty} a_n x^n$ of the initial value problem

$$(1+2x^2)y'' + 10xy' + 8y = 0, \quad y(0) = 2, \quad y'(0) = -3. \quad (7.2.29)$$

Solution Since $\alpha = 2, \beta = 10$, and $\gamma = 8$ in (7.2.29),

$$p(n) = 2n(n-1) + 10n + 8 = 2(n+2)^2.$$

Therefore

$$a_{n+2} = -2 \frac{(n+2)^2}{(n+2)(n+1)} a_n = -2 \frac{n+2}{n+1} a_n, \quad n \geq 0.$$

Writing this equation separately for $n = 2m$ and $n = 2m+1$ yields

$$a_{2m+2} = -2 \frac{(2m+2)}{2m+1} a_{2m} = -4 \frac{m+1}{2m+1} a_{2m}, \quad m \geq 0 \quad (7.2.30)$$

and

$$a_{2m+3} = -2 \frac{2m+3}{2m+2} a_{2m+1} = -\frac{2m+3}{m+1} a_{2m+1}, \quad m \geq 0. \quad (7.2.31)$$

Starting with $a_0 = y(0) = 2$, we compute a_2, a_4 , and a_6 from (7.2.30):

$$\begin{aligned} a_2 &= -4 \frac{1}{1} 2 = -8, \\ a_4 &= -4 \frac{2}{3} (-8) = \frac{64}{3}, \\ a_6 &= -4 \frac{3}{5} \left(\frac{64}{3} \right) = -\frac{256}{5}. \end{aligned}$$

Starting with $a_1 = y'(0) = -3$, we compute a_3 , a_5 and a_7 from (7.2.31):

$$\begin{aligned} a_3 &= -\frac{3}{1}(-3) = 9, \\ a_5 &= -\frac{5}{2}9 = -\frac{45}{2}, \\ a_7 &= -\frac{7}{3}\left(-\frac{45}{2}\right) = \frac{105}{2}. \end{aligned}$$

Therefore the solution of (7.2.29) is

$$y = 2 - 3x - 8x^2 + 9x^3 + \frac{64}{3}x^4 - \frac{45}{2}x^5 - \frac{256}{5}x^6 + \frac{105}{2}x^7 + \cdots.$$

Section 7.3

Example 7.3.1 Find the coefficients $a_0, \dots, a_7$ in the series solution $y = \sum_{n=0}^{\infty} a_n x^n$ of the initial value problem

$$(1 + x + 2x^2)y'' + (1 + 7x)y' + 2y = 0, \quad y(0) = -1, \quad y'(0) = -2. \quad (7.3.2)$$

Solution Here

$$Ly = (1 + x + 2x^2)y'' + (1 + 7x)y' + 2y.$$

The zeros $(-1 \pm i\sqrt{7})/4$ of $P_0(x) = 1 + x + 2x^2$ have absolute value $1/\sqrt{2}$, so Theorem 7.2.1 implies that the series solution converges to the solution of (7.3.2) on $(-1/\sqrt{2}, 1/\sqrt{2})$. Since

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2},$$

$$\begin{aligned} Ly &= \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1} + 2 \sum_{n=2}^{\infty} n(n-1) a_n x^n \\ &\quad + \sum_{n=1}^{\infty} n a_n x^{n-1} + 7 \sum_{n=1}^{\infty} n a_n x^n + 2 \sum_{n=0}^{\infty} a_n x^n. \end{aligned}$$

Shifting indices so the general term in each series is a constant multiple of x^n yields

$$\begin{aligned} Ly &= \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n + \sum_{n=0}^{\infty} (n+1) n a_{n+1} x^n + 2 \sum_{n=0}^{\infty} n(n-1) a_n x^n \\ &\quad + \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n + 7 \sum_{n=0}^{\infty} n a_n x^n + 2 \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} b_n x^n, \end{aligned}$$

where

$$b_n = (n+2)(n+1) a_{n+2} + (n+1)^2 a_{n+1} + (n+2)(2n+1) a_n.$$

Therefore $y = \sum_{n=0}^{\infty} a_n x^n$ is a solution of $Ly = 0$ if and only if

$$a_{n+2} = -\frac{n+1}{n+2} a_{n+1} - \frac{2n+1}{n+1} a_n, \quad n \geq 0. \quad (7.3.3)$$

From the initial conditions in (7.3.2), $a_0 = y(0) = -1$ and $a_1 = y'(0) = -2$. Setting $n = 0$ in (7.3.3) yields

$$a_2 = -\frac{1}{2}a_1 - a_0 = -\frac{1}{2}(-2) - (-1) = 2.$$

Setting $n = 1$ in (7.3.3) yields

$$a_3 = -\frac{2}{3}a_2 - \frac{3}{2}a_1 = -\frac{2}{3}(2) - \frac{3}{2}(-2) = \frac{5}{3}.$$

We leave it to you to compute a_4, a_5, a_6, a_7 from (7.3.3) and show that

$$y = -1 - 2x + 2x^2 + \frac{5}{3}x^3 - \frac{55}{12}x^4 + \frac{3}{4}x^5 + \frac{61}{8}x^6 - \frac{443}{56}x^7 + \cdots.$$

We also leave it to you (Exercise 7.3.13) to verify numerically that the Taylor polynomials $T_N(x) = \sum_{n=0}^N a_n x^n$ converge to the solution of (7.3.2) on $(-1/\sqrt{2}, 1/\sqrt{2})$.

Example 7.3.2 Find the coefficients $a_0, \dots, a_5$ in the series solution

$$y = \sum_{n=0}^{\infty} a_n (x+1)^n$$

of the initial value problem

$$(3+x)y'' + (1+2x)y' - (2-x)y = 0, \quad y(-1) = 2, \quad y'(-1) = -3. \quad (7.3.4)$$

Solution Since the desired series is in powers of $x+1$ we rewrite the differential equation in (7.3.4) as $Ly = 0$, with

$$Ly = (2 + (x+1))y'' - (1 - 2(x+1))y' - (3 - (x+1))y.$$

Since

$$y = \sum_{n=0}^{\infty} a_n (x+1)^n, \quad y' = \sum_{n=1}^{\infty} n a_n (x+1)^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x+1)^{n-2},$$

$$\begin{aligned} Ly &= 2 \sum_{n=2}^{\infty} n(n-1) a_n (x+1)^{n-2} + \sum_{n=2}^{\infty} n(n-1) a_n (x+1)^{n-1} \\ &\quad - \sum_{n=1}^{\infty} n a_n (x+1)^{n-1} + 2 \sum_{n=1}^{\infty} n a_n (x+1)^n \\ &\quad - 3 \sum_{n=0}^{\infty} a_n (x+1)^n + \sum_{n=0}^{\infty} a_n (x+1)^{n+1}. \end{aligned}$$

Shifting indices so that the general term in each series is a constant multiple of $(x+1)^n$ yields

$$\begin{aligned} Ly &= 2 \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}(x+1)^n + \sum_{n=0}^{\infty} (n+1)na_{n+1}(x+1)^n \\ &\quad - \sum_{n=0}^{\infty} (n+1)a_{n+1}(x+1)^n + \sum_{n=0}^{\infty} (2n-3)a_n(x+1)^n + \sum_{n=1}^{\infty} a_{n-1}(x+1)^n \\ &= \sum_{n=0}^{\infty} b_n(x+1)^n, \end{aligned}$$

where

$$b_0 = 4a_2 - a_1 - 3a_0$$

and

$$b_n = 2(n+2)(n+1)a_{n+2} + (n^2 - 1)a_{n+1} + (2n-3)a_n + a_{n-1}, \quad n \geq 1.$$

Therefore $y = \sum_{n=0}^{\infty} a_n(x+1)^n$ is a solution of $Ly = 0$ if and only if

$$a_2 = \frac{1}{4}(a_1 + 3a_0) \quad (7.3.5)$$

and

$$a_{n+2} = -\frac{1}{2(n+2)(n+1)} [(n^2 - 1)a_{n+1} + (2n-3)a_n + a_{n-1}], \quad n \geq 1. \quad (7.3.6)$$

From the initial conditions in (7.3.4), $a_0 = y(-1) = 2$ and $a_1 = y'(-1) = -3$. We leave it to you to compute $a_2, \dots, a_5$ with (7.3.5) and (7.3.6) and show that the solution of (7.3.4) is

$$y = 2 - 3(x+1) + \frac{3}{4}(x+1)^2 - \frac{5}{12}(x+1)^3 + \frac{7}{48}(x+1)^4 - \frac{1}{60}(x+1)^5 + \dots$$

We also leave it to you (Exercise 7.3.14) to verify numerically that the Taylor polynomials $T_N(x) = \sum_{n=0}^N a_n(x+1)^n$ converge to the solution of (7.3.4) on the interval of convergence of the power series solution.

Example 7.3.3 Find the coefficients $a_0, \dots, a_5$ in the series solution $y = \sum_{n=0}^{\infty} a_n x^n$ of the initial value problem

$$y'' + 3xy' + (4 + 2x^2)y = 0, \quad y(0) = 2, \quad y'(0) = -3. \quad (7.3.7)$$

Solution Here

$$Ly = y'' + 3xy' + (4 + 2x^2)y.$$

Since

$$y = \sum_{n=0}^{\infty} a_n x^n, \quad y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2},$$

$$Ly = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + 3 \sum_{n=1}^{\infty} n a_n x^n + 4 \sum_{n=0}^{\infty} a_n x^n + 2 \sum_{n=0}^{\infty} a_n x^{n+2}.$$

Shifting indices so that the general term in each series is a constant multiple of x^n yields

$$Ly = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}x^n + \sum_{n=0}^{\infty} (3n+4)a_nx^n + 2 \sum_{n=2}^{\infty} a_{n-2}x^n = \sum_{n=0}^{\infty} b_nx^n$$

where

$$b_0 = 2a_2 + 4a_0, \quad b_1 = 6a_3 + 7a_1,$$

and

$$b_n = (n+2)(n+1)a_{n+2} + (3n+4)a_n + 2a_{n-2}, \quad n \geq 2.$$

Therefore $y = \sum_{n=0}^{\infty} a_nx^n$ is a solution of $Ly = 0$ if and only if

$$a_2 = -2a_0, \quad a_3 = -\frac{7}{6}a_1, \quad (7.3.8)$$

and

$$a_{n+2} = -\frac{1}{(n+2)(n+1)} [(3n+4)a_n + 2a_{n-2}], \quad n \geq 2. \quad (7.3.9)$$

From the initial conditions in (7.3.7), $a_0 = y(0) = 2$ and $a_1 = y'(0) = -3$. We leave it to you to compute $a_2, \dots, a_5$ with (7.3.8) and (7.3.9) and show that the solution of (7.3.7) is

$$y = 2 - 3x - 4x^2 + \frac{7}{2}x^3 + 3x^4 - \frac{79}{40}x^5 + \dots$$

We also leave it to you (Exercise 7.3.15) to verify numerically that the Taylor polynomials $T_N(x) = \sum_{n=0}^N a_nx^n$ converge to the solution of (7.3.7) on the interval of convergence of the power series solution.

Section 7.4

Example 7.4.1 Bessel's equation,

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \quad (7.4.4)$$

has the singular point $x_0 = 0$. Since this equation is of the form (7.4.3) with $x_0 = 0$, $A(x) = 1$, $B(x) = 1$, and $C(x) = x^2 - \nu^2$, it follows that $x_0 = 0$ is a regular singular point of (7.4.4).

Example 7.4.2 Legendre's equation,

$$(1 - x^2)y'' - 2xy' + \alpha(\alpha + 1)y = 0, \quad (7.4.5)$$

has the singular points $x_0 = \pm 1$. Multiplying through by $1 - x$ yields

$$(x - 1)^2(x + 1)y'' + 2x(x - 1)y' - \alpha(\alpha + 1)(x - 1)y = 0,$$

which is of the form (7.4.3) with $x_0 = 1$, $A(x) = x + 1$, $B(x) = 2x$, and $C(x) = -\alpha(\alpha + 1)(x - 1)$. Therefore $x_0 = 1$ is a regular singular point of (7.4.5). We leave it to you to show that $x_0 = -1$ is also a regular singular point of (7.4.5).

Example 7.4.3 The equation

$$x^3y'' + xy' + y = 0$$

has an irregular singular point at $x_0 = 0$. (Verify.)

Example 7.4.4 Find the general solution of

$$x^2 y'' - xy' - 8y = 0 \quad (7.4.8)$$

on $(0, \infty)$.

Solution The indicial polynomial of (7.4.8) is

$$p(r) = r(r-1) - r - 8 = (r-4)(r+2).$$

Therefore $y_1 = x^4$ and $y_2 = x^{-2}$ are solutions of (7.4.8) on $(0, \infty)$, and its general solution on $(0, \infty)$ is

$$y = c_1 x^4 + \frac{c_2}{x^2}.$$

Example 7.4.5 Find the general solution of

$$6x^2 y'' + 5xy' - y = 0 \quad (7.4.9)$$

on $(0, \infty)$.

Solution The indicial polynomial of (7.4.9) is

$$p(r) = 6r(r-1) + 5r - 1 = (2r-1)(3r+1).$$

Therefore the general solution of (7.4.9) on $(0, \infty)$ is

$$y = c_1 x^{1/2} + c_2 x^{-1/3}. \blacksquare$$

Example 7.4.6 Find the general solution of

$$x^2 y'' - 5xy' + 9y = 0 \quad (7.4.15)$$

on $(0, \infty)$.

Solution The indicial polynomial of (7.4.15) is

$$p(r) = r(r-1) - 5r + 9 = (r-3)^2.$$

Therefore the general solution of (7.4.15) on $(0, \infty)$ is

$$y = x^3(c_1 + c_2 \ln x).$$

Example 7.4.7 Find the general solution of

$$x^2 y'' + 3xy' + 2y = 0 \quad (7.4.16)$$

on $(0, \infty)$.

Solution The indicial polynomial of (7.4.16) is

$$p(r) = r(r-1) + 3r + 2 = (r+1)^2 + 1.$$

The roots of the indicial equation are $r = -1 \pm i$ and the general solution of (7.4.16) on $(0, \infty)$ is

$$y = \frac{1}{x} [c_1 \cos(\ln x) + c_2 \sin(\ln x)].$$

Section 7.5

Example 7.5.1 Find a fundamental set of Frobenius solutions of

$$2x^2(1+x+x^2)y'' + x(9+11x+11x^2)y' + (6+10x+7x^2)y = 0. \quad (7.5.16)$$

Compute just the first six coefficients $a_0, \dots, a_5$ in each solution.

Solution For the given equation, the polynomials defined in Theorem 7.5.2 are

$$\begin{aligned} p_0(r) &= 2r(r-1) + 9r + 6 = (2r+3)(r+2), \\ p_1(r) &= 2r(r-1) + 11r + 10 = (2r+5)(r+2), \\ p_2(r) &= 2r(r-1) + 11r + 7 = (2r+7)(r+1). \end{aligned}$$

The zeros of the indicial polynomial p_0 are $r_1 = -3/2$ and $r_2 = -2$, so $r_1 - r_2 = 1/2$. Therefore Theorem 7.5.3 implies that

$$y_1 = x^{-3/2} \sum_{n=0}^{\infty} a_n(-3/2)x^n \quad \text{and} \quad y_2 = x^{-2} \sum_{n=0}^{\infty} a_n(-2)x^n \quad (7.5.17)$$

form a fundamental set of Frobenius solutions of (7.5.16). To find the coefficients in these series, we use the recurrence relation of Theorem 7.5.2; thus,

$$\begin{aligned} a_0(r) &= 1, \\ a_1(r) &= -\frac{p_1(r)}{p_0(r+1)} = -\frac{(2r+5)(r+2)}{(2r+5)(r+3)} = -\frac{r+2}{r+3}, \\ a_n(r) &= -\frac{p_1(n+r-1)a_{n-1} + p_2(n+r-2)a_{n-2}}{p_0(n+r)} \\ &= -\frac{(n+r+1)(2n+2r+3)a_{n-1}(r) + (n+r-1)(2n+2r+3)a_{n-2}(r)}{(n+r+2)(2n+2r+3)} \\ &= -\frac{(n+r+1)a_{n-1}(r) + (n+r-1)a_{n-2}(r)}{n+r+2}, \quad n \geq 2. \end{aligned}$$

Setting $r = -3/2$ in these equations yields

$$\begin{aligned} a_0(-3/2) &= 1, \\ a_1(-3/2) &= -1/3, \\ a_n(-3/2) &= -\frac{(2n-1)a_{n-1}(-3/2) + (2n-5)a_{n-2}(-3/2)}{2n+1}, \quad n \geq 2, \end{aligned} \quad (7.5.18)$$

and setting $r = -2$ yields

$$\begin{aligned} a_0(-2) &= 1, \\ a_1(-2) &= 0, \\ a_n(-2) &= -\frac{(n-1)a_{n-1}(-2) + (n-3)a_{n-2}(-2)}{n}, \quad n \geq 2. \end{aligned} \quad (7.5.19)$$

Calculating with (7.5.18) and (7.5.19) and substituting the results into (7.5.17) yields the fundamental set of Frobenius solutions

$$\begin{aligned} y_1 &= x^{-3/2} \left(1 - \frac{1}{3}x + \frac{2}{5}x^2 - \frac{5}{21}x^3 + \frac{7}{135}x^4 + \frac{76}{1155}x^5 + \cdots \right), \\ y_2 &= x^{-2} \left(1 + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{8}x^4 + \frac{1}{30}x^5 + \cdots \right). \end{aligned}$$

Example 7.5.2 Find a fundamental set of Frobenius solutions of

$$x^2(3+x)y'' + 5x(1+x)y' - (1-4x)y = 0. \quad (7.5.21)$$

Give explicit formulas for the coefficients in the solutions.

Solution For this equation, the polynomials defined in Theorem 7.5.2 are

$$\begin{aligned} p_0(r) &= 3r(r-1) + 5r - 1 = (3r-1)(r+1), \\ p_1(r) &= r(r-1) + 5r + 4 = (r+2)^2, \\ p_2(r) &= 0. \end{aligned}$$

The zeros of the indicial polynomial p_0 are $r_1 = 1/3$ and $r_2 = -1$, so $r_1 - r_2 = 4/3$. Therefore Theorem 7.5.3 implies that

$$y_1 = x^{1/3} \sum_{n=0}^{\infty} a_n(1/3)x^n \quad \text{and} \quad y_2 = x^{-1} \sum_{n=0}^{\infty} a_n(-1)x^n$$

form a fundamental set of Frobenius solutions of (7.5.21). To find the coefficients in these series, we use the recurrence relations (7.5.20); thus,

$$\begin{aligned} a_0(r) &= 1, \\ a_n(r) &= -\frac{p_1(n+r-1)}{p_0(n+r)} a_{n-1}(r) \\ &= -\frac{(n+r+1)^2}{(3n+3r-1)(n+r+1)} a_{n-1}(r) \\ &= -\frac{n+r+1}{3n+3r-1} a_{n-1}(r), \quad n \geq 1. \end{aligned} \quad (7.5.22)$$

Setting $r = 1/3$ in (7.5.22) yields

$$\begin{aligned} a_0(1/3) &= 1, \\ a_n(1/3) &= -\frac{3n+4}{9n} a_{n-1}(1/3), \quad n \geq 1. \end{aligned}$$

By using the product notation introduced in Section 7.2 and proceeding as we did in the examples in that section yields

$$a_n(1/3) = \frac{(-1)^n \prod_{j=1}^n (3j+4)}{9^n n!}, \quad n \geq 0.$$

Therefore

$$y_1 = x^{1/3} \sum_{n=0}^{\infty} \frac{(-1)^n \prod_{j=1}^n (3j+4)}{9^n n!} x^n$$

is a Frobenius solution of (7.5.21).

Setting $r = -1$ in (7.5.22) yields

$$\begin{aligned} a_0(-1) &= 1, \\ a_n(-1) &= -\frac{n}{3n-4} a_{n-1}(-1), \quad n \geq 1, \end{aligned}$$

so

$$a_n(-1) = \frac{(-1)^n n!}{\prod_{j=1}^n (3j-4)}.$$

Therefore

$$y_2 = x^{-1} \sum_{n=0}^{\infty} \frac{(-1)^n n!}{\prod_{j=1}^n (3j-4)} x^n$$

is a Frobenius solution of (7.5.21), and $\{y_1, y_2\}$ is a fundamental set of solutions. ■

Example 7.5.3 Find a fundamental set of Frobenius solutions of

$$x^2(2-x^2)y'' - x(3+4x^2)y' + (2-2x^2)y = 0. \quad (7.5.25)$$

Give explicit formulas for the coefficients in the solutions.

Solution For this equation, the polynomials defined in Theorem 7.5.2 are

$$\begin{aligned} p_0(r) &= 2r(r-1) - 3r + 2 = (r-2)(2r-1), \\ p_1(r) &= 0 \\ p_2(r) &= -[r(r-1) + 4r + 2] = -(r+1)(r+2). \end{aligned}$$

The zeros of the indicial polynomial p_0 are $r_1 = 2$ and $r_2 = 1/2$, so $r_1 - r_2 = 3/2$. Therefore Theorem 7.5.3 implies that

$$y_1 = x^2 \sum_{m=0}^{\infty} a_{2m}(2)x^{2m} \quad \text{and} \quad y_2 = x^{1/2} \sum_{m=0}^{\infty} a_{2m}(1/2)x^{2m}$$

form a fundamental set of Frobenius solutions of (7.5.25). To find the coefficients in these series, we use the recurrence relation (7.5.24); thus,

$$\begin{aligned} a_0(r) &= 1, \\ a_{2m}(r) &= -\frac{p_2(2m+r-2)}{p_0(2m+r)} a_{2m-2}(r) \\ &= \frac{(2m+r)(2m+r-1)}{(2m+r-2)(4m+2r-1)} a_{2m-2}(r), \quad m \geq 1. \end{aligned} \quad (7.5.26)$$

Setting $r = 2$ in (7.5.26) yields

$$\begin{aligned} a_0(2) &= 1, \\ a_{2m}(2) &= \frac{(m+1)(2m+1)}{m(4m+3)} a_{2m-2}(2), \quad m \geq 1, \end{aligned}$$

so

$$a_{2m}(2) = (m+1) \prod_{j=1}^m \frac{2j+1}{4j+3}.$$

Therefore

$$y_1 = x^2 \sum_{m=0}^{\infty} (m+1) \left(\prod_{j=1}^m \frac{2j+1}{4j+3} \right) x^{2m}$$

is a Frobenius solution of (7.5.25).

Setting $r = 1/2$ in (7.5.26) yields

$$\begin{aligned} a_0(1/2) &= 1, \\ a_{2m}(1/2) &= \frac{(4m-1)(4m+1)}{8m(4m-3)} a_{2m-2}(1/2), \quad m \geq 1, \end{aligned}$$

so

$$a_{2m}(1/2) = \frac{1}{8^m m!} \prod_{j=1}^m \frac{(4j-1)(4j+1)}{4j-3}.$$

Therefore

$$y_2 = x^{1/2} \sum_{m=0}^{\infty} \frac{1}{8^m m!} \left(\prod_{j=1}^m \frac{(4j-1)(4j+1)}{4j-3} \right) x^{2m}$$

is a Frobenius solution of (7.5.25) and $\{y_1, y_2\}$ is a fundamental set of solutions.

Section 7.6

Example 7.6.1 Find a fundamental set of solutions of

$$x^2(1-2x+x^2)y'' - x(3+x)y' + (4+x)y = 0. \quad (7.6.9)$$

Compute just the terms involving x^{n+r_1} , where $0 \leq n \leq 4$ and r_1 is the root of the indicial equation.

Solution For the given equation, the polynomials defined in Theorem 7.6.1 are

$$\begin{aligned} p_0(r) &= r(r-1) - 3r + 4 = (r-2)^2, \\ p_1(r) &= -2r(r-1) - r + 1 = -(r-1)(2r+1), \\ p_2(r) &= r(r-1). \end{aligned}$$

Since $r_1 = 2$ is a repeated root of the indicial polynomial p_0 , Theorem 7.6.2 implies that

$$y_1 = x^2 \sum_{n=0}^{\infty} a_n(2)x^n \quad \text{and} \quad y_2 = y_1 \ln x + x^2 \sum_{n=1}^{\infty} a'_n(2)x^n \quad (7.6.10)$$

form a fundamental set of Frobenius solutions of (7.6.9). To find the coefficients in these series, we use the recurrence formulas from Theorem 7.6.1:

$$\begin{aligned} a_0(r) &= 1, \\ a_1(r) &= -\frac{p_1(r)}{p_0(r+1)} = \frac{(r-1)(2r+1)}{(r-1)^2} = \frac{2r+1}{r-1}, \\ a_n(r) &= -\frac{p_1(n+r-1)a_{n-1}(r) + p_2(n+r-2)a_{n-2}(r)}{p_0(n+r)} \\ &= \frac{(n+r-2)[(2n+2r-1)a_{n-1}(r) - (n+r-3)a_{n-2}(r)]}{(n+r-2)^2} \\ &= \frac{(2n+2r-1)}{(n+r-2)}a_{n-1}(r) - \frac{(n+r-3)}{(n+r-2)}a_{n-2}(r), \quad n \geq 2. \end{aligned} \quad (7.6.11)$$

Differentiating yields

$$\begin{aligned} a'_1(r) &= -\frac{3}{(r-1)^2}, \\ a'_n(r) &= \frac{2n+2r-1}{n+r-2}a'_{n-1}(r) - \frac{n+r-3}{n+r-2}a'_{n-2}(r) \\ &\quad - \frac{3}{(n+r-2)^2}a_{n-1}(r) - \frac{1}{(n+r-2)^2}a_{n-2}(r), \quad n \geq 2. \end{aligned} \quad (7.6.12)$$

Setting $r = 2$ in (7.6.11) and (7.6.12) yields

$$\begin{aligned} a_0(2) &= 1, \\ a_1(2) &= 5, \\ a_n(2) &= \frac{(2n+3)}{n}a_{n-1}(2) - \frac{(n-1)}{n}a_{n-2}(2), \quad n \geq 2 \end{aligned} \tag{7.6.13}$$

and

$$\begin{aligned} a'_1(2) &= -3, \\ a'_n(2) &= \frac{2n+3}{n}a'_{n-1}(2) - \frac{n-1}{n}a'_{n-2}(2) - \frac{3}{n^2}a_{n-1}(2) - \frac{1}{n^2}a_{n-2}(2), \quad n \geq 2. \end{aligned} \tag{7.6.14}$$

Computing recursively with (7.6.13) and (7.6.14) yields

$$a_0(2) = 1, a_1(2) = 5, a_2(2) = 17, a_3(2) = \frac{143}{3}, a_4(2) = \frac{355}{3},$$

and

$$a'_1(2) = -3, a'_2(2) = -\frac{29}{2}, a'_3(2) = -\frac{859}{18}, a'_4(2) = -\frac{4693}{36}.$$

Substituting these coefficients into (7.6.10) yields

$$y_1 = x^2 \left(1 + 5x + 17x^2 + \frac{143}{3}x^3 + \frac{355}{3}x^4 + \cdots \right)$$

and

$$y_2 = y_1 \ln x - x^3 \left(3 + \frac{29}{2}x + \frac{859}{18}x^2 + \frac{4693}{36}x^3 + \cdots \right). \blacksquare$$

Example 7.6.2 Find a fundamental set of Frobenius solutions of

$$2x^2(2+x)y'' + 5x^2y' + (1+x)y = 0. \tag{7.6.15}$$

Give explicit formulas for the coefficients in the solutions.

Solution For the given equation, the polynomials defined in Theorem 7.6.1 are

$$\begin{aligned} p_0(r) &= 4r(r-1) + 1 = (2r-1)^2, \\ p_1(r) &= 2r(r-1) + 5r + 1 = (r+1)(2r+1), \\ p_2(r) &= 0. \end{aligned}$$

Since $r_1 = 1/2$ is a repeated zero of the indicial polynomial p_0 , Theorem 7.6.2 implies that

$$y_1 = x^{1/2} \sum_{n=0}^{\infty} a_n(1/2)x^n \tag{7.6.16}$$

and

$$y_2 = y_1 \ln x + x^{1/2} \sum_{n=1}^{\infty} a'_n(1/2)x^n \tag{7.6.17}$$

form a fundamental set of Frobenius solutions of (7.6.15). Since $p_2 \equiv 0$, the recurrence formulas in Theorem 7.6.1 reduce to

$$\begin{aligned} a_0(r) &= 1, \\ a_n(r) &= -\frac{p_1(n+r-1)}{p_0(n+r)}a_{n-1}(r), \\ &= -\frac{(n+r)(2n+2r-1)}{(2n+2r-1)^2}a_{n-1}(r), \\ &= -\frac{n+r}{2n+2r-1}a_{n-1}(r), \quad n \geq 1. \end{aligned}$$

We leave it to you to show that

$$a_n(r) = (-1)^n \prod_{j=1}^n \frac{j+r}{2j+2r-1}, \quad n \geq 0. \quad (7.6.18)$$

Setting $r = 1/2$ yields

$$\begin{aligned} a_n(1/2) &= (-1)^n \prod_{j=1}^n \frac{j+1/2}{2j} = (-1)^n \prod_{j=1}^n \frac{2j+1}{4j}, \\ &= \frac{(-1)^n \prod_{j=1}^n (2j+1)}{4^n n!}, \quad n \geq 0. \end{aligned} \quad (7.6.19)$$

Substituting this into (7.6.16) yields

$$y_1 = x^{1/2} \sum_{n=0}^{\infty} \frac{(-1)^n \prod_{j=1}^n (2j+1)}{4^n n!} x^n.$$

To obtain y_2 in (7.6.17), we must compute $a'_n(1/2)$ for $n = 1, 2, \dots$. We'll do this by logarithmic differentiation. From (7.6.18),

$$|a_n(r)| = \prod_{j=1}^n \frac{|j+r|}{|2j+2r-1|}, \quad n \geq 1.$$

Therefore

$$\ln |a_n(r)| = \sum_{j=1}^n (\ln |j+r| - \ln |2j+2r-1|).$$

Differentiating with respect to r yields

$$\frac{a'_n(r)}{a_n(r)} = \sum_{j=1}^n \left(\frac{1}{j+r} - \frac{2}{2j+2r-1} \right).$$

Therefore

$$a'_n(r) = a_n(r) \sum_{j=1}^n \left(\frac{1}{j+r} - \frac{2}{2j+2r-1} \right).$$

Setting $r = 1/2$ here and recalling (7.6.19) yields

$$a'_n(1/2) = \frac{(-1)^n \prod_{j=1}^n (2j+1)}{4^n n!} \left(\sum_{j=1}^n \frac{1}{j+1/2} - \sum_{j=1}^n \frac{1}{j} \right). \quad (7.6.20)$$

Since

$$\frac{1}{j+1/2} - \frac{1}{j} = \frac{j-j-1/2}{j(j+1/2)} = -\frac{1}{j(2j+1)},$$

(7.6.20) can be rewritten as

$$a'_n(1/2) = -\frac{(-1)^n \prod_{j=1}^n (2j+1)}{4^n n!} \sum_{j=1}^n \frac{1}{j(2j+1)}.$$

Therefore, from (7.6.17),

$$y_2 = y_1 \ln x - x^{1/2} \sum_{n=1}^{\infty} \frac{(-1)^n \prod_{j=1}^n (2j+1)}{4^n n!} \left(\sum_{j=1}^n \frac{1}{j(2j+1)} \right) x^n.$$

Example 7.6.3 Find a fundamental set of Frobenius solutions of

$$x^2(2-x^2)y'' - 2x(1+2x^2)y' + (2-2x^2)y = 0. \quad (7.6.21)$$

Give explicit formulas for the coefficients in the solutions.

Solution For (7.6.21), the polynomials defined in Theorem 7.6.1 are

$$\begin{aligned} p_0(r) &= 2r(r-1) - 2r + 2 = 2(r-1)^2, \\ p_1(r) &= 0, \\ p_2(r) &= -r(r-1) - 4r - 2 = -(r+1)(r+2). \end{aligned}$$

As in Section 7.5, since $p_1 \equiv 0$, the recurrence formulas of Theorem 7.6.1 imply that $a_n(r) = 0$ if n is odd, and

$$\begin{aligned} a_0(r) &= 1, \\ a_{2m}(r) &= -\frac{p_2(2m+r-2)}{p_0(2m+r)} a_{2m-2}(r) \\ &= \frac{(2m+r-1)(2m+r)}{2(2m+r-1)^2} a_{2m-2}(r) \\ &= \frac{2m+r}{2(2m+r-1)} a_{2m-2}(r), \quad m \geq 1. \end{aligned}$$

Since $r_1 = 1$ is a repeated root of the indicial polynomial p_0 , Theorem 7.6.2 implies that

$$y_1 = x \sum_{m=0}^{\infty} a_{2m}(1) x^{2m} \quad (7.6.22)$$

and

$$y_2 = y_1 \ln x + x \sum_{m=1}^{\infty} a'_{2m}(1) x^{2m} \quad (7.6.23)$$

form a fundamental set of Frobenius solutions of (7.6.21). We leave it to you to show that

$$a_{2m}(r) = \frac{1}{2^m} \prod_{j=1}^m \frac{2j+r}{2j+r-1}. \quad (7.6.24)$$

Setting $r = 1$ yields

$$a_{2m}(1) = \frac{1}{2^m} \prod_{j=1}^m \frac{2j+1}{2j} = \frac{\prod_{j=1}^m (2j+1)}{4^m m!}, \quad (7.6.25)$$

and substituting this into (7.6.22) yields

$$y_1 = x \sum_{m=0}^{\infty} \frac{\prod_{j=1}^m (2j+1)}{4^m m!} x^{2m}.$$

To obtain y_2 in (7.6.23), we must compute $a'_{2m}(1)$ for $m = 1, 2, \dots$. Again we use logarithmic differentiation. From (7.6.24),

$$|a_{2m}(r)| = \frac{1}{2^m} \prod_{j=1}^m \frac{|2j+r|}{|2j+r-1|}.$$

Taking logarithms yields

$$\ln |a_{2m}(r)| = -m \ln 2 + \sum_{j=1}^m (\ln |2j+r| - \ln |2j+r-1|).$$

Differentiating with respect to r yields

$$\frac{a'_{2m}(r)}{a_{2m}(r)} = \sum_{j=1}^m \left(\frac{1}{2j+r} - \frac{1}{2j+r-1} \right).$$

Therefore

$$a'_{2m}(r) = a_{2m}(r) \sum_{j=1}^m \left(\frac{1}{2j+r} - \frac{1}{2j+r-1} \right).$$

Setting $r = 1$ and recalling (7.6.25) yields

$$a'_{2m}(1) = \frac{\prod_{j=1}^m (2j+1)}{4^m m!} \sum_{j=1}^m \left(\frac{1}{2j+1} - \frac{1}{2j} \right). \quad (7.6.26)$$

Since

$$\frac{1}{2j+1} - \frac{1}{2j} = -\frac{1}{2j(2j+1)},$$

(7.6.26) can be rewritten as

$$a'_{2m}(1) = -\frac{\prod_{j=1}^m (2j+1)}{2 \cdot 4^m m!} \sum_{j=1}^m \frac{1}{j(2j+1)}.$$

Substituting this into (7.6.23) yields

$$y_2 = y_1 \ln x - \frac{x}{2} \sum_{m=1}^{\infty} \frac{\prod_{j=1}^m (2j+1)}{4^m m!} \left(\sum_{j=1}^m \frac{1}{j(2j+1)} \right) x^{2m}. \blacksquare$$

Example 7.6.4 Find a fundamental set of Frobenius solutions of

$$x^2 y'' - x(5-x)y' + (9-4x)y = 0. \quad (7.6.27)$$

Give explicit formulas for the coefficients in the solutions.

Solution For (7.6.27) the polynomials defined in Theorem 7.6.1 are

$$\begin{aligned} p_0(r) &= r(r-1) - 5r + 9 = (r-3)^2, \\ p_1(r) &= r - 4, \\ p_2(r) &= 0. \end{aligned}$$

Since $r_1 = 3$ is a repeated zero of the indicial polynomial p_0 , Theorem 7.6.2 implies that

$$y_1 = x^3 \sum_{n=0}^{\infty} a_n(3)x^n \quad (7.6.28)$$

and

$$y_2 = y_1 \ln x + x^3 \sum_{n=1}^{\infty} a'_n(3)x^n \quad (7.6.29)$$

are linearly independent Frobenius solutions of (7.6.27). To find the coefficients in (7.6.28) we use the recurrence formulas

$$\begin{aligned} a_0(r) &= 1, \\ a_n(r) &= -\frac{p_1(n+r-1)}{p_0(n+r)} a_{n-1}(r) \\ &= -\frac{n+r-5}{(n+r-3)^2} a_{n-1}(r), \quad n \geq 1. \end{aligned}$$

We leave it to you to show that

$$a_n(r) = (-1)^n \prod_{j=1}^n \frac{j+r-5}{(j+r-3)^2}. \quad (7.6.30)$$

Setting $r = 3$ here yields

$$a_n(3) = (-1)^n \prod_{j=1}^n \frac{j-2}{j^2},$$

so $a_1(3) = 1$ and $a_n(3) = 0$ if $n \geq 2$. Substituting these coefficients into (7.6.28) yields

$$y_1 = x^3(1+x).$$

To obtain y_2 in (7.6.29) we must compute $a'_n(3)$ for $n = 1, 2, \dots$. Let's first try logarithmic differentiation. From (7.6.30),

$$|a_n(r)| = \prod_{j=1}^n \frac{|j+r-5|}{|j+r-3|^2}, \quad n \geq 1,$$

so

$$\ln |a_n(r)| = \sum_{j=1}^n (\ln |j+r-5| - 2 \ln |j+r-3|).$$

Differentiating with respect to r yields

$$\frac{a'_n(r)}{a_n(r)} = \sum_{j=1}^n \left(\frac{1}{j+r-5} - \frac{2}{j+r-3} \right).$$

Therefore

$$a'_n(r) = a_n(r) \sum_{j=1}^n \left(\frac{1}{j+r-5} - \frac{2}{j+r-3} \right). \quad (7.6.31)$$

However, we can't simply set $r = 3$ here if $n \geq 2$, since the bracketed expression in the sum corresponding to $j = 2$ contains the term $1/(r-3)$. In fact, since $a_n(3) = 0$ for $n \geq 2$, the formula (7.6.31) for $a'_n(r)$ is actually an indeterminate form at $r = 3$.

We overcome this difficulty as follows. From (7.6.30) with $n = 1$,

$$a_1(r) = -\frac{r-4}{(r-2)^2}.$$

Therefore

$$a'_1(r) = \frac{r-6}{(r-2)^3},$$

so

$$a'_1(3) = -3. \quad (7.6.32)$$

From (7.6.30) with $n \geq 2$,

$$a_n(r) = (-1)^n(r-4)(r-3) \frac{\prod_{j=3}^n (j+r-5)}{\prod_{j=1}^n (j+r-3)^2} = (r-3)c_n(r),$$

where

$$c_n(r) = (-1)^n(r-4) \frac{\prod_{j=3}^n (j+r-5)}{\prod_{j=1}^n (j+r-3)^2}, \quad n \geq 2.$$

Therefore

$$a'_n(r) = c_n(r) + (r-3)c'_n(r), \quad n \geq 2,$$

which implies that $a'_n(3) = c_n(3)$ if $n \geq 3$. We leave it to you to verify that

$$a'_n(3) = c_n(3) = \frac{(-1)^{n+1}}{n(n-1)n!}, \quad n \geq 2.$$

Substituting this and (7.6.32) into (7.6.29) yields

$$y_2 = x^3(1+x) \ln x - 3x^4 - x^3 \sum_{n=2}^{\infty} \frac{(-1)^n}{n(n-1)n!} x^n.$$

Section 7.7

Example 7.7.1 Find a fundamental set of Frobenius solutions of

$$2x^2(2+x)y'' - x(4-7x)y' - (5-3x)y = 0.$$

Give explicit formulas for the coefficients in the solutions.

Solution For the given equation, the polynomials defined in Theorem 7.7.1 are

$$\begin{aligned} p_0(r) &= 4r(r-1) - 4r - 5 = (2r+1)(2r-5), \\ p_1(r) &= 2r(r-1) + 7r + 3 = (r+1)(2r+3). \end{aligned}$$

The roots of the indicial equation are $r_1 = 5/2$ and $r_2 = -1/2$, so $k = r_1 - r_2 = 3$. Therefore Theorem 7.7.1 implies that

$$y_1 = x^{5/2} \sum_{n=0}^{\infty} a_n(5/2)x^n \quad (7.7.11)$$

and

$$y_2 = x^{-1/2} \sum_{n=0}^2 a_n(-1/2)x^n + C \left(y_1 \ln x + x^{5/2} \sum_{n=1}^{\infty} a'_n(5/2)x^n \right) \quad (7.7.12)$$

(with C as in (7.7.4)) form a fundamental set of solutions of $Ly = 0$. The recurrence formula (7.7.2) is

$$\begin{aligned} a_0(r) &= 1, \\ a_n(r) &= -\frac{p_1(n+r-1)}{p_0(n+r)} a_{n-1}(r) \\ &= -\frac{(n+r)(2n+2r+1)}{(2n+2r+1)(2n+2r-5)} a_{n-1}(r), \\ &= -\frac{n+r}{2n+2r-5} a_{n-1}(r), \quad n \geq 1, \end{aligned} \quad (7.7.13)$$

which implies that

$$a_n(r) = (-1)^n \prod_{j=1}^n \frac{j+r}{2j+2r-5}, \quad n \geq 0. \quad (7.7.14)$$

Therefore

$$a_n(5/2) = \frac{(-1)^n \prod_{j=1}^n (2j+5)}{4^n n!}. \quad (7.7.15)$$

Substituting this into (7.7.11) yields

$$y_1 = x^{5/2} \sum_{n=0}^{\infty} \frac{(-1)^n \prod_{j=1}^n (2j+5)}{4^n n!} x^n.$$

To compute the coefficients $a_0(-1/2)$, $a_1(-1/2)$ and $a_2(-1/2)$ in y_2 , we set $r = -1/2$ in (7.7.13) and apply the resulting recurrence formula for $n = 1, 2$; thus,

$$\begin{aligned} a_0(-1/2) &= 1, \\ a_n(-1/2) &= -\frac{2n-1}{4(n-3)} a_{n-1}(-1/2), \quad n = 1, 2. \end{aligned}$$

The last formula yields

$$a_1(-1/2) = 1/8 \quad \text{and} \quad a_2(-1/2) = 3/32.$$

Substituting $r_1 = 5/2$, $r_2 = -1/2$, $k = 3$, and $\alpha_0 = 4$ into (7.7.4) yields $C = -15/128$. Therefore, from (7.7.12),

$$y_2 = x^{-1/2} \left(1 + \frac{1}{8}x + \frac{3}{32}x^2 \right) - \frac{15}{128} \left(y_1 \ln x + x^{5/2} \sum_{n=1}^{\infty} a'_n(5/2)x^n \right). \quad (7.7.16)$$

We use logarithmic differentiation to obtain $a'_n(r)$. From (7.7.14),

$$|a_n(r)| = \prod_{j=1}^n \frac{|j+r|}{|2j+2r-5|}, \quad n \geq 1.$$

Therefore

$$\ln |a_n(r)| = \sum_{j=1}^n (\ln |j+r| - \ln |2j+2r-5|).$$

Differentiating with respect to r yields

$$\frac{a'_n(r)}{a_n(r)} = \sum_{j=1}^n \left(\frac{1}{j+r} - \frac{2}{2j+2r-5} \right).$$

Therefore

$$a'_n(r) = a_n(r) \sum_{j=1}^n \left(\frac{1}{j+r} - \frac{2}{2j+2r-5} \right).$$

Setting $r = 5/2$ here and recalling (7.7.15) yields

$$a'_n(5/2) = \frac{(-1)^n \prod_{j=1}^n (2j+5)}{4^n n!} \sum_{j=1}^n \left(\frac{1}{j+5/2} - \frac{1}{j} \right). \quad (7.7.17)$$

Since

$$\frac{1}{j+5/2} - \frac{1}{j} = -\frac{5}{j(2j+5)},$$

we can rewrite (7.7.17) as

$$a'_n(5/2) = -5 \frac{(-1)^n \prod_{j=1}^n (2j+5)}{4^n n!} \left(\sum_{j=1}^n \frac{1}{j(2j+5)} \right).$$

Substituting this into (7.7.16) yields

$$\begin{aligned} y_2 &= x^{-1/2} \left(1 + \frac{1}{8}x + \frac{3}{32}x^2 \right) - \frac{15}{128}y_1 \ln x \\ &\quad + \frac{75}{128}x^{5/2} \sum_{n=1}^{\infty} \frac{(-1)^n \prod_{j=1}^n (2j+5)}{4^n n!} \left(\sum_{j=1}^n \frac{1}{j(2j+5)} \right) x^n. \blacksquare \end{aligned}$$

Example 7.7.2 Find a fundamental set of Frobenius solutions of

$$x^2(1-2x)y'' + x(8-9x)y' + (6-3x)y = 0.$$

Give explicit formulas for the coefficients in the solutions.

Solution For the given equation, the polynomials defined in Theorem 7.7.1 are

$$\begin{aligned} p_0(r) &= r(r-1) + 8r + 6 = (r+1)(r+6) \\ p_1(r) &= -2r(r-1) - 9r - 3 = -(r+3)(2r+1). \end{aligned}$$

The roots of the indicial equation are $r_1 = -1$ and $r_2 = -6$, so $k = r_1 - r_2 = 5$. Therefore Theorem 7.7.1 implies that

$$y_1 = x^{-1} \sum_{n=0}^{\infty} a_n(-1)x^n \quad (7.7.18)$$

and

$$y_2 = x^{-6} \sum_{n=0}^4 a_n(-6)x^n + C \left(y_1 \ln x + x^{-1} \sum_{n=1}^{\infty} a'_n(-1)x^n \right) \quad (7.7.19)$$

(with C as in (7.7.4)) form a fundamental set of solutions of $Ly = 0$. The recurrence formula (7.7.2) is

$$\begin{aligned} a_0(r) &= 1, \\ a_n(r) &= -\frac{p_1(n+r-1)}{p_0(n+r)} a_{n-1}(r) \\ &= \frac{(n+r+2)(2n+2r-1)}{(n+r+1)(n+r+6)} a_{n-1}(r), \quad n \geq 1, \end{aligned} \quad (7.7.20)$$

which implies that

$$\begin{aligned} a_n(r) &= \prod_{j=1}^n \frac{(j+r+2)(2j+2r-1)}{(j+r+1)(j+r+6)} \\ &= \left(\prod_{j=1}^n \frac{j+r+2}{j+r+1} \right) \left(\prod_{j=1}^n \frac{2j+2r-1}{j+r+6} \right). \end{aligned} \quad (7.7.21)$$

Since

$$\prod_{j=1}^n \frac{j+r+2}{j+r+1} = \frac{(r+3)(r+4) \cdots (n+r+2)}{(r+2)(r+3) \cdots (n+r+1)} = \frac{n+r+2}{r+2}$$

because of cancellations, (7.7.21) simplifies to

$$a_n(r) = \frac{n+r+2}{r+2} \prod_{j=1}^n \frac{2j+2r-1}{j+r+6}.$$

Therefore

$$a_n(-1) = (n+1) \prod_{j=1}^n \frac{2j-3}{j+5}.$$

Substituting this into (7.7.18) yields

$$y_1 = x^{-1} \sum_{n=0}^{\infty} (n+1) \left(\prod_{j=1}^n \frac{2j-3}{j+5} \right) x^n.$$

To compute the coefficients $a_0(-6), \dots, a_4(-6)$ in y_2 , we set $r = -6$ in (7.7.20) and apply the resulting recurrence formula for $n = 1, 2, 3, 4$; thus,

$$\begin{aligned} a_0(-6) &= 1, \\ a_n(-6) &= \frac{(n-4)(2n-13)}{n(n-5)} a_{n-1}(-6), \quad n = 1, 2, 3, 4. \end{aligned}$$

The last formula yields

$$a_1(-6) = -\frac{33}{4}, \quad a_2(-6) = \frac{99}{4}, \quad a_3(-6) = -\frac{231}{8}, \quad a_4(-6) = 0.$$

Since $a_4(-6) = 0$, (7.7.4) implies that the constant C in (7.7.19) is zero. Therefore (7.7.19) reduces to

$$y_2 = x^{-6} \left(1 - \frac{33}{4}x + \frac{99}{4}x^2 - \frac{231}{8}x^3 \right). \quad \blacksquare$$

Example 7.7.3 Find a fundamental set of Frobenius solutions of

$$x^2(1+x^2)y'' + x(3+10x^2)y' - (15-14x^2)y = 0.$$

Give explicit formulas for the coefficients in the solutions.

Solution For the given equation, the polynomials defined in Theorem 7.7.2 are

$$\begin{aligned} p_0(r) &= r(r-1) + 3r - 15 = (r-3)(r+5) \\ p_2(r) &= r(r-1) + 10r + 14 = (r+2)(r+7). \end{aligned}$$

The roots of the indicial equation are $r_1 = 3$ and $r_2 = -5$, so $k = (r_1 - r_2)/2 = 4$. Therefore Theorem 7.7.2 implies that

$$y_1 = x^3 \sum_{m=0}^{\infty} a_{2m}(3)x^{2m} \quad (7.7.25)$$

and

$$y_2 = x^{-5} \sum_{m=0}^3 a_{2m}(-5)x^{2m} + C \left(y_1 \ln x + x^3 \sum_{m=1}^{\infty} a'_{2m}(3)x^{2m} \right)$$

(with C as in (7.7.23)) form a fundamental set of solutions of $Ly = 0$. The recurrence formula (7.7.22) is

$$\begin{aligned} a_0(r) &= 1, \\ a_{2m}(r) &= -\frac{p_2(2m+r-2)}{p_0(2m+r)} a_{2m-2}(r) \\ &= -\frac{(2m+r)(2m+r+5)}{(2m+r-3)(2m+r+5)} a_{2m-2}(r) \\ &= -\frac{2m+r}{2m+r-3} a_{2m-2}(r), \quad m \geq 1, \end{aligned} \quad (7.7.26)$$

which implies that

$$a_{2m}(r) = (-1)^m \prod_{j=1}^m \frac{2j+r}{2j+r-3}, \quad m \geq 0. \quad (7.7.27)$$

Therefore

$$a_{2m}(3) = \frac{(-1)^m \prod_{j=1}^m (2j+3)}{2^m m!}. \quad (7.7.28)$$

Substituting this into (7.7.25) yields

$$y_1 = x^3 \sum_{m=0}^{\infty} \frac{(-1)^m \prod_{j=1}^m (2j+3)}{2^m m!} x^{2m}.$$

To compute the coefficients $a_2(-5)$, $a_4(-5)$, and $a_6(-5)$ in y_2 , we set $r = -5$ in (7.7.26) and apply the resulting recurrence formula for $m = 1, 2, 3$; thus,

$$a_{2m}(-5) = -\frac{2m-5}{2(m-4)} a_{2m-2}(-5), \quad m = 1, 2, 3.$$

This yields

$$a_2(-5) = -\frac{1}{2}, \quad a_4(-5) = \frac{1}{8}, \quad a_6(-5) = \frac{1}{16}.$$

Substituting $r_1 = 3$, $r_2 = -5$, $k = 4$, and $\alpha_0 = 1$ into (7.7.23) yields $C = -3/16$. Therefore, from (7.7.24),

$$y_2 = x^{-5} \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{16}x^6 \right) - \frac{3}{16} \left(y_1 \ln x + x^3 \sum_{m=1}^{\infty} a'_{2m}(3)x^{2m} \right). \quad (7.7.29)$$

To obtain $a'_{2m}(r)$ we use logarithmic differentiation. From (7.7.27),

$$|a_{2m}(r)| = \prod_{j=1}^m \frac{|2j+r|}{|2j+r-3|}, \quad m \geq 1.$$

Therefore

$$\ln |a_{2m}(r)| = \sum_{j=1}^m (\ln |2j+r| - \ln |2j+r-3|).$$

Differentiating with respect to r yields

$$\frac{a'_{2m}(r)}{a_{2m}(r)} = \sum_{j=1}^m \left(\frac{1}{2j+r} - \frac{1}{2j+r-3} \right).$$

Therefore

$$a'_{2m}(r) = a_{2m}(r) \sum_{j=1}^m \left(\frac{1}{2j+r} - \frac{1}{2j+r-3} \right).$$

Setting $r = 3$ here and recalling (7.7.28) yields

$$a'_{2m}(3) = \frac{(-1)^m \prod_{j=1}^m (2j+3)}{2^m m!} \sum_{j=1}^m \left(\frac{1}{2j+3} - \frac{1}{2j} \right). \quad (7.7.30)$$

Since

$$\frac{1}{2j+3} - \frac{1}{2j} = -\frac{3}{2j(2j+3)},$$

we can rewrite (7.7.30) as

$$a'_{2m}(3) = -\frac{3}{2} \frac{(-1)^m \prod_{j=1}^m (2j+3)}{2^m m!} \left(\sum_{j=1}^m \frac{1}{j(2j+3)} \right).$$

Substituting this into (7.7.29) yields

$$\begin{aligned} y_2 &= x^{-5} \left(1 - \frac{1}{2}x^2 + \frac{1}{8}x^4 + \frac{1}{16}x^6 \right) - \frac{3}{16} y_1 \ln x \\ &\quad + \frac{9}{32} x^3 \sum_{m=1}^{\infty} \frac{(-1)^m \prod_{j=1}^m (2j+3)}{2^m m!} \left(\sum_{j=1}^m \frac{1}{j(2j+3)} \right) x^{2m}. \end{aligned}$$

Example 7.7.4 Find a fundamental set of Frobenius solutions of

$$x^2(1-2x^2)y'' + x(7-13x^2)y' - 14x^2y = 0.$$

Give explicit formulas for the coefficients in the solutions.

Solution For the given equation, the polynomials defined in Theorem 7.7.2 are

$$\begin{aligned} p_0(r) &= r(r-1) + 7r = r(r+6), \\ p_2(r) &= -2r(r-1) - 13r - 14 = -(r+2)(2r+7). \end{aligned}$$

The roots of the indicial equation are $r_1 = 0$ and $r_2 = -6$, so $k = (r_1 - r_2)/2 = 3$. Therefore Theorem 7.7.2 implies that

$$y_1 = \sum_{m=0}^{\infty} a_{2m}(0)x^{2m}, \quad (7.7.31)$$

and

$$y_2 = x^{-6} \sum_{m=0}^2 a_{2m}(-6)x^{2m} + C \left(y_1 \ln x + \sum_{m=1}^{\infty} a'_{2m}(0)x^{2m} \right) \quad (7.7.32)$$

(with C as in (7.7.23)) form a fundamental set of solutions of $Ly = 0$. The recurrence formulas (7.7.22) are

$$\begin{aligned} a_0(r) &= 1, \\ a_{2m}(r) &= -\frac{p_2(2m+r-2)}{p_0(2m+r)} a_{2m-2}(r) \\ &= \frac{(2m+r)(4m+2r+3)}{(2m+r)(2m+r+6)} a_{2m-2}(r) \\ &= \frac{4m+2r+3}{2m+r+6} a_{2m-2}(r), \quad m \geq 1, \end{aligned} \quad (7.7.33)$$

which implies that

$$a_{2m}(r) = \prod_{j=1}^m \frac{4j+2r+3}{2j+r+6}.$$

Setting $r = 0$ yields

$$a_{2m}(0) = 6 \frac{\prod_{j=1}^m (4j+3)}{2^m (m+3)!}.$$

Substituting this into (7.7.31) yields

$$y_1 = 6 \sum_{m=0}^{\infty} \frac{\prod_{j=1}^m (4j+3)}{2^m (m+3)!} x^{2m}.$$

To compute the coefficients $a_0(-6)$, $a_2(-6)$, and $a_4(-6)$ in y_2 , we set $r = -6$ in (7.7.33) and apply the resulting recurrence formula for $m = 1, 2$; thus,

$$\begin{aligned} a_0(-6) &= 1, \\ a_{2m}(-6) &= \frac{4m-9}{2m} a_{2m-2}(-6), \quad m = 1, 2. \end{aligned}$$

The last formula yields

$$a_2(-6) = -\frac{5}{2} \quad \text{and} \quad a_4(-6) = \frac{5}{8}.$$

Since $p_2(-2) = 0$, the constant C in (7.7.23) is zero. Therefore (7.7.32) reduces to

$$y_2 = x^{-6} \left(1 - \frac{5}{2}x^2 + \frac{5}{8}x^4 \right).$$

Section 8.1

Example 8.1.1 Find the Laplace transform of $f(t) = 1$.

Solution From (8.1.2) with $f(t) = 1$,

$$F(s) = \int_0^{\infty} e^{-st} dt = \lim_{T \rightarrow \infty} \int_0^T e^{-st} dt.$$

If $s \neq 0$ then

$$\int_0^T e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_0^T = \frac{1 - e^{-sT}}{s}. \quad (8.1.3)$$

Therefore

$$\lim_{T \rightarrow \infty} \int_0^T e^{-st} dt = \begin{cases} \frac{1}{s}, & s > 0, \\ \infty, & s < 0. \end{cases} \quad (8.1.4)$$

If $s = 0$ the integrand reduces to the constant 1, and

$$\lim_{T \rightarrow \infty} \int_0^T 1 dt = \lim_{T \rightarrow \infty} T = \infty.$$

Therefore $F(0)$ is undefined, and

$$F(s) = \int_0^{\infty} e^{-st} dt = \frac{1}{s}, \quad s > 0.$$

This result can be written in operator notation as

$$\mathcal{L}(1) = \frac{1}{s}, \quad s > 0,$$

or as the transform pair

$$1 \leftrightarrow \frac{1}{s}, \quad s > 0.$$

Remark: It is convenient to combine the steps of integrating from 0 to T and letting $T \rightarrow \infty$. Therefore, instead of writing (8.1.3) and (8.1.4) as separate steps we write

$$\int_0^{\infty} e^{-st} dt = -\frac{1}{s} e^{-st} \Big|_0^{\infty} = \begin{cases} \frac{1}{s}, & s > 0, \\ \infty, & s < 0. \end{cases}$$

We'll follow this practice throughout this chapter.

Example 8.1.2 Find the Laplace transform of $f(t) = t$.

Solution From (8.1.2) with $f(t) = t$,

$$F(s) = \int_0^{\infty} e^{-st} t dt. \quad (8.1.5)$$

If $s \neq 0$, integrating by parts yields

$$\begin{aligned} \int_0^\infty e^{-st} t \, dt &= -\frac{te^{-st}}{s} \Big|_0^\infty + \frac{1}{s} \int_0^\infty e^{-st} \, dt = -\left[\frac{t}{s} + \frac{1}{s^2} \right] e^{-st} \Big|_0^\infty \\ &= \begin{cases} \frac{1}{s^2}, & s > 0, \\ \infty, & s < 0. \end{cases} \end{aligned}$$

If $s = 0$, the integral in (8.1.5) becomes

$$\int_0^\infty t \, dt = \frac{t^2}{2} \Big|_0^\infty = \infty.$$

Therefore $F(0)$ is undefined and

$$F(s) = \frac{1}{s^2}, \quad s > 0.$$

This result can also be written as

$$\mathcal{L}(t) = \frac{1}{s^2}, \quad s > 0,$$

or as the transform pair

$$t \leftrightarrow \frac{1}{s^2}, \quad s > 0.$$

Example 8.1.3 Find the Laplace transform of $f(t) = e^{at}$, where a is a constant.

Solution From (8.1.2) with $f(t) = e^{at}$,

$$F(s) = \int_0^\infty e^{-st} e^{at} \, dt.$$

Combining the exponentials yields

$$F(s) = \int_0^\infty e^{-(s-a)t} \, dt.$$

However, we know from Example 8.1.1 that

$$\int_0^\infty e^{-st} \, dt = \frac{1}{s}, \quad s > 0.$$

Replacing s by $s - a$ here shows that

$$F(s) = \frac{1}{s - a}, \quad s > a.$$

This can also be written as

$$\mathcal{L}(e^{at}) = \frac{1}{s - a}, \quad s > a, \quad \text{or} \quad e^{at} \leftrightarrow \frac{1}{s - a}, \quad s > a.$$

Example 8.1.4 Find the Laplace transforms of $f(t) = \sin \omega t$ and $g(t) = \cos \omega t$, where ω is a constant.

Solution Define

$$F(s) = \int_0^{\infty} e^{-st} \sin \omega t \, dt \quad (8.1.6)$$

and

$$G(s) = \int_0^{\infty} e^{-st} \cos \omega t \, dt. \quad (8.1.7)$$

If $s > 0$, integrating (8.1.6) by parts yields

$$F(s) = -\frac{e^{-st}}{s} \sin \omega t \Big|_0^{\infty} + \frac{\omega}{s} \int_0^{\infty} e^{-st} \cos \omega t \, dt,$$

so

$$F(s) = \frac{\omega}{s} G(s). \quad (8.1.8)$$

If $s > 0$, integrating (8.1.7) by parts yields

$$G(s) = -\frac{e^{-st} \cos \omega t}{s} \Big|_0^{\infty} - \frac{\omega}{s} \int_0^{\infty} e^{-st} \sin \omega t \, dt,$$

so

$$G(s) = \frac{1}{s} - \frac{\omega}{s} F(s).$$

Now substitute from (8.1.8) into this to obtain

$$G(s) = \frac{1}{s} - \frac{\omega^2}{s^2} G(s).$$

Solving this for $G(s)$ yields

$$G(s) = \frac{s}{s^2 + \omega^2}, \quad s > 0.$$

This and (8.1.8) imply that

$$F(s) = \frac{\omega}{s^2 + \omega^2}, \quad s > 0.$$

Example 8.1.5 Use the table of Laplace transforms to find $\mathcal{L}(t^3 e^{4t})$.

Solution The table includes the transform pair

$$t^n e^{at} \leftrightarrow \frac{n!}{(s-a)^{n+1}}.$$

Setting $n = 3$ and $a = 4$ here yields

$$\mathcal{L}(t^3 e^{4t}) = \frac{3!}{(s-4)^4} = \frac{6}{(s-4)^4}. \quad \blacksquare$$

Example 8.1.6 Use Theorem 8.1.2 and the known Laplace transform

$$\mathcal{L}(e^{at}) = \frac{1}{s-a}$$

to find $\mathcal{L}(\cosh bt)$ ($b \neq 0$).

Solution By definition,

$$\cosh bt = \frac{e^{bt} + e^{-bt}}{2}.$$

Therefore

$$\begin{aligned}\mathcal{L}(\cosh bt) &= \mathcal{L}\left(\frac{1}{2}e^{bt} + \frac{1}{2}e^{-bt}\right) \\ &= \frac{1}{2}\mathcal{L}(e^{bt}) + \frac{1}{2}\mathcal{L}(e^{-bt}) \quad (\text{linearity property}) \\ &= \frac{1}{2} \frac{1}{s-b} + \frac{1}{2} \frac{1}{s+b},\end{aligned}\tag{8.1.9}$$

where the first transform on the right is defined for $s > b$ and the second for $s > -b$; hence, both are defined for $s > |b|$. Simplifying the last expression in (8.1.9) yields

$$\mathcal{L}(\cosh bt) = \frac{s}{s^2 - b^2}, \quad s > |b|.$$

Example 8.1.7 Use Theorem 8.1.3 and the known Laplace transforms of 1, t , $\cos \omega t$, and $\sin \omega t$ to find

$$\mathcal{L}(e^{at}), \quad \mathcal{L}(te^{at}), \quad \mathcal{L}(e^{\lambda t} \sin \omega t), \text{ and } \mathcal{L}(e^{\lambda t} \cos \omega t).$$

Solution In the following table the known transform pairs are listed on the left and the required transform pairs listed on the right are obtained by applying Theorem 8.1.3.

$f(t) \leftrightarrow F(s)$	$e^{at}f(t) \leftrightarrow F(s-a)$
$1 \leftrightarrow \frac{1}{s}, \quad s > 0$	$e^{at} \leftrightarrow \frac{1}{(s-a)}, \quad s > a$
$t \leftrightarrow \frac{1}{s^2}, \quad s > 0$	$te^{at} \leftrightarrow \frac{1}{(s-a)^2}, \quad s > a$
$\sin \omega t \leftrightarrow \frac{\omega}{s^2 + \omega^2}, \quad s > 0$	$e^{\lambda t} \sin \omega t \leftrightarrow \frac{\omega}{(s-\lambda)^2 + \omega^2}, \quad s > \lambda$
$\cos \omega t \leftrightarrow \frac{s}{s^2 + \omega^2}, \quad s > 0$	$e^{\lambda t} \cos \omega t \leftrightarrow \frac{s-\lambda}{(s-\lambda)^2 + \omega^2}, \quad s > \lambda$

Example 8.1.8 If f is bounded on some interval $[t_0, \infty)$, say

$$|f(t)| \leq M, \quad t \geq t_0,$$

then (8.1.14) holds with $s_0 = 0$, so f is of exponential order zero. Thus, for example, $\sin \omega t$ and $\cos \omega t$ are of exponential order zero, and Theorem 8.1.6 implies that $\mathcal{L}(\sin \omega t)$ and $\mathcal{L}(\cos \omega t)$ exist for $s > 0$. This is consistent with the conclusion of Example 8.1.4.

Example 8.1.9 It can be shown that if $\lim_{t \rightarrow \infty} e^{-s_0 t} f(t)$ exists and is finite then f is of exponential order s_0 (Exercise 8.1.9). If α is any real number and $s_0 > 0$ then $f(t) = t^\alpha$ is of exponential order s_0 , since

$$\lim_{t \rightarrow \infty} e^{-s_0 t} t^\alpha = 0,$$

by L'Hôpital's rule. If $\alpha \geq 0$, f is also continuous on $[0, \infty)$. Therefore Exercise 8.1.9 and Theorem 8.1.6 imply that $\mathcal{L}(t^\alpha)$ exists for $s > s_0$. However, since s_0 is an arbitrary positive number, this really implies that $\mathcal{L}(t^\alpha)$ exists for all $s > 0$. This is consistent with the results of Example 8.1.2 and Exercises 8.1.6 and 8.1.8.

Example 8.1.10 Find the Laplace transform of the piecewise continuous function

$$f(t) = \begin{cases} 1, & 0 \leq t < 1, \\ -3e^{-t}, & t \geq 1. \end{cases}$$

Solution Since f is defined by different formulas on $[0, 1)$ and $[1, \infty)$, we write

$$F(s) = \int_0^{\infty} e^{-st} f(t) dt = \int_0^1 e^{-st}(1) dt + \int_1^{\infty} e^{-st}(-3e^{-t}) dt.$$

Since

$$\int_0^1 e^{-st} dt = \begin{cases} \frac{1 - e^{-s}}{s}, & s \neq 0, \\ 1, & s = 0, \end{cases}$$

and

$$\int_1^{\infty} e^{-st}(-3e^{-t}) dt = -3 \int_1^{\infty} e^{-(s+1)t} dt = -\frac{3e^{-(s+1)}}{s+1}, \quad s > -1,$$

it follows that

$$F(s) = \begin{cases} \frac{1 - e^{-s}}{s} - 3\frac{e^{-(s+1)}}{s+1}, & s > -1, s \neq 0, \\ 1 - \frac{3}{e}, & s = 0. \end{cases}$$

This is consistent with Theorem 8.1.6, since

$$|f(t)| \leq 3e^{-t}, \quad t \geq 1,$$

and therefore f is of exponential order $s_0 = -1$.

Example 8.1.11 We stated earlier that

$$\int_0^{\infty} e^{-st} e^{t^2} dt = \infty$$

for all s , so Theorem 8.1.6 implies that $f(t) = e^{t^2}$ is not of exponential order, since

$$\lim_{t \rightarrow \infty} \frac{e^{t^2}}{Me^{s_0 t}} = \lim_{t \rightarrow \infty} \frac{1}{M} e^{t^2 - s_0 t} = \infty,$$

so

$$e^{t^2} > Me^{s_0 t}$$

for sufficiently large values of t , for any choice of M and s_0 (Exercise 8.1.3).

Section 8.2

Example 8.2.1 Use the table of Laplace transforms to find

$$\text{(a)} \mathcal{L}^{-1}\left(\frac{1}{s^2 - 1}\right) \quad \text{and} \quad \text{(b)} \mathcal{L}^{-1}\left(\frac{s}{s^2 + 9}\right).$$

Solution (a) Setting $b = 1$ in the transform pair

$$\sinh bt \leftrightarrow \frac{b}{s^2 - b^2}$$

shows that

$$\mathcal{L}^{-1}\left(\frac{1}{s^2 - 1}\right) = \sinh t.$$

Solution (b) Setting $\omega = 3$ in the transform pair

$$\cos \omega t \leftrightarrow \frac{s}{s^2 + \omega^2}$$

shows that

$$\mathcal{L}^{-1}\left(\frac{s}{s^2 + 9}\right) = \cos 3t. \quad \blacksquare$$

Example 8.2.2 Find

$$\mathcal{L}^{-1}\left(\frac{8}{s+5} + \frac{7}{s^2+3}\right).$$

Solution From the table of Laplace transforms in Section 8.8,

$$e^{at} \leftrightarrow \frac{1}{s-a} \quad \text{and} \quad \sin \omega t \leftrightarrow \frac{\omega}{s^2 + \omega^2}.$$

Theorem 8.2.1 with $a = -5$ and $\omega = \sqrt{3}$ yields

$$\begin{aligned} \mathcal{L}^{-1}\left(\frac{8}{s+5} + \frac{7}{s^2+3}\right) &= 8\mathcal{L}^{-1}\left(\frac{1}{s+5}\right) + 7\mathcal{L}^{-1}\left(\frac{1}{s^2+3}\right) \\ &= 8\mathcal{L}^{-1}\left(\frac{1}{s+5}\right) + \frac{7}{\sqrt{3}}\mathcal{L}^{-1}\left(\frac{\sqrt{3}}{s^2+3}\right) \\ &= 8e^{-5t} + \frac{7}{\sqrt{3}}\sin \sqrt{3}t. \end{aligned}$$

Example 8.2.3 Find

$$\mathcal{L}^{-1}\left(\frac{3s+8}{s^2+2s+5}\right).$$

Solution Completing the square in the denominator yields

$$\frac{3s+8}{s^2+2s+5} = \frac{3s+8}{(s+1)^2+4}.$$

Because of the form of the denominator, we consider the transform pairs

$$e^{-t} \cos 2t \leftrightarrow \frac{s+1}{(s+1)^2+4} \quad \text{and} \quad e^{-t} \sin 2t \leftrightarrow \frac{2}{(s+1)^2+4},$$

and write

$$\begin{aligned}
 \mathcal{L}^{-1}\left(\frac{3s+8}{(s+1)^2+4}\right) &= \mathcal{L}^{-1}\left(\frac{3s+3}{(s+1)^2+4}\right) + \mathcal{L}^{-1}\left(\frac{5}{(s+1)^2+4}\right) \\
 &= 3\mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+4}\right) + \frac{5}{2}\mathcal{L}^{-1}\left(\frac{2}{(s+1)^2+4}\right) \\
 &= e^{-t}(3\cos 2t + \frac{5}{2}\sin 2t).
 \end{aligned}$$

Example 8.2.4 Find the inverse Laplace transform of

$$F(s) = \frac{3s+2}{s^2-3s+2}. \quad (8.2.1)$$

Solution (METHOD 1) Factoring the denominator in (8.2.1) yields

$$F(s) = \frac{3s+2}{(s-1)(s-2)}. \quad (8.2.2)$$

The form for the partial fraction expansion is

$$\frac{3s+2}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}. \quad (8.2.3)$$

Multiplying this by $(s-1)(s-2)$ yields

$$3s+2 = (s-2)A + (s-1)B.$$

Setting $s = 2$ yields $B = 8$ and setting $s = 1$ yields $A = -5$. Therefore

$$F(s) = -\frac{5}{s-1} + \frac{8}{s-2}$$

and

$$\mathcal{L}^{-1}(F) = -5\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) + 8\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) = -5e^t + 8e^{2t}.$$

Solution (METHOD 2) We don't really have to multiply (8.2.3) by $(s-1)(s-2)$ to compute A and B . We can obtain A by simply ignoring the factor $s-1$ in the denominator of (8.2.2) and setting $s = 1$ elsewhere; thus,

$$A = \left.\frac{3s+2}{s-2}\right|_{s=1} = \frac{3 \cdot 1 + 2}{1-2} = -5. \quad (8.2.4)$$

Similarly, we can obtain B by ignoring the factor $s-2$ in the denominator of (8.2.2) and setting $s = 2$ elsewhere; thus,

$$B = \left.\frac{3s+2}{s-1}\right|_{s=2} = \frac{3 \cdot 2 + 2}{2-1} = 8. \quad (8.2.5)$$

To justify this, we observe that multiplying (8.2.3) by $s-1$ yields

$$\frac{3s+2}{s-2} = A + (s-1)\frac{B}{s-2},$$

and setting $s = 1$ leads to (8.2.4). Similarly, multiplying (8.2.3) by $s - 2$ yields

$$\frac{3s + 2}{s - 1} = (s - 2) \frac{A}{s - 1} + B$$

and setting $s = 2$ leads to (8.2.5). (It isn't necessary to write the last two equations. We wrote them only to justify the shortcut procedure indicated in (8.2.4) and (8.2.5).) ■

Example 8.2.5 Find the inverse Laplace transform of

$$F(s) = \frac{6 + (s + 1)(s^2 - 5s + 11)}{s(s - 1)(s - 2)(s + 1)}. \quad (8.2.7)$$

Solution The partial fraction expansion of (8.2.7) is of the form

$$F(s) = \frac{A}{s} + \frac{B}{s - 1} + \frac{C}{s - 2} + \frac{D}{s + 1}. \quad (8.2.8)$$

To find A , we ignore the factor s in the denominator of (8.2.7) and set $s = 0$ elsewhere. This yields

$$A = \frac{6 + (1)(11)}{(-1)(-2)(1)} = \frac{17}{2}.$$

Similarly, the other coefficients are given by

$$B = \frac{6 + (2)(7)}{(1)(-1)(2)} = -10,$$

$$C = \frac{6 + 3(5)}{2(1)(3)} = \frac{7}{2},$$

and

$$D = \frac{6}{(-1)(-2)(-3)} = -1.$$

Therefore

$$F(s) = \frac{17}{2} \frac{1}{s} - \frac{10}{s - 1} + \frac{7}{2} \frac{1}{s - 2} - \frac{1}{s + 1}$$

and

$$\begin{aligned} \mathcal{L}^{-1}(F) &= \frac{17}{2} \mathcal{L}^{-1}\left(\frac{1}{s}\right) - 10 \mathcal{L}^{-1}\left(\frac{1}{s - 1}\right) + \frac{7}{2} \mathcal{L}^{-1}\left(\frac{1}{s - 2}\right) - \mathcal{L}^{-1}\left(\frac{1}{s + 1}\right) \\ &= \frac{17}{2} - 10e^t + \frac{7}{2}e^{2t} - e^{-t}. \end{aligned}$$

Remark: We didn't "multiply out" the numerator in (8.2.7) before computing the coefficients in (8.2.8), since it wouldn't simplify the computations.

Example 8.2.6 Find the inverse Laplace transform of

$$F(s) = \frac{8 - (s + 2)(4s + 10)}{(s + 1)(s + 2)^2}. \quad (8.2.9)$$

Solution The form for the partial fraction expansion is

$$F(s) = \frac{A}{s+1} + \frac{B}{s+2} + \frac{C}{(s+2)^2}. \quad (8.2.10)$$

Because of the repeated factor $(s+2)^2$ in (8.2.9), Heaviside's method doesn't work. Instead, we find a common denominator in (8.2.10). This yields

$$F(s) = \frac{A(s+2)^2 + B(s+1)(s+2) + C(s+1)}{(s+1)(s+2)^2}. \quad (8.2.11)$$

If (8.2.9) and (8.2.11) are to be equivalent, then

$$A(s+2)^2 + B(s+1)(s+2) + C(s+1) = 8 - (s+2)(4s+10). \quad (8.2.12)$$

The two sides of this equation are polynomials of degree two. From a theorem of algebra, they will be equal for all s if they are equal for any three distinct values of s . We may determine A , B and C by choosing convenient values of s .

The left side of (8.2.12) suggests that we take $s = -2$ to obtain $C = -8$, and $s = -1$ to obtain $A = 2$. We can now choose any third value of s to determine B . Taking $s = 0$ yields $4A + 2B + C = -12$. Since $A = 2$ and $C = -8$ this implies that $B = -6$. Therefore

$$F(s) = \frac{2}{s+1} - \frac{6}{s+2} - \frac{8}{(s+2)^2}$$

and

$$\begin{aligned} \mathcal{L}^{-1}(F) &= 2\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) - 6\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) - 8\mathcal{L}^{-1}\left(\frac{1}{(s+2)^2}\right) \\ &= 2e^{-t} - 6e^{-2t} - 8te^{-2t}. \end{aligned}$$

Example 8.2.7 Find the inverse Laplace transform of

$$F(s) = \frac{s^2 - 5s + 7}{(s+2)^3}.$$

Solution The form for the partial fraction expansion is

$$F(s) = \frac{A}{s+2} + \frac{B}{(s+2)^2} + \frac{C}{(s+2)^3}.$$

The easiest way to obtain A , B , and C is to expand the numerator in powers of $s+2$. This yields

$$s^2 - 5s + 7 = [(s+2) - 2]^2 - 5[(s+2) - 2] + 7 = (s+2)^2 - 9(s+2) + 21.$$

Therefore

$$\begin{aligned} F(s) &= \frac{(s+2)^2 - 9(s+2) + 21}{(s+2)^3} \\ &= \frac{1}{s+2} - \frac{9}{(s+2)^2} + \frac{21}{(s+2)^3} \end{aligned}$$

and

$$\begin{aligned}\mathcal{L}^{-1}(F) &= \mathcal{L}^{-1}\left(\frac{1}{s+2}\right) - 9\mathcal{L}^{-1}\left(\frac{1}{(s+2)^2}\right) + \frac{21}{2}\mathcal{L}^{-1}\left(\frac{2}{(s+2)^3}\right) \\ &= e^{-2t}\left(1 - 9t + \frac{21}{2}t^2\right).\end{aligned}$$

Example 8.2.8 Find the inverse Laplace transform of

$$F(s) = \frac{1 - s(5 + 3s)}{s[(s + 1)^2 + 1]}. \quad (8.2.13)$$

Solution One form for the partial fraction expansion of F is

$$F(s) = \frac{A}{s} + \frac{Bs + C}{(s + 1)^2 + 1}. \quad (8.2.14)$$

However, we see from the table of Laplace transforms that the inverse transform of the second fraction on the right of (8.2.14) will be a linear combination of the inverse transforms

$$e^{-t} \cos t \quad \text{and} \quad e^{-t} \sin t$$

of

$$\frac{s + 1}{(s + 1)^2 + 1} \quad \text{and} \quad \frac{1}{(s + 1)^2 + 1}$$

respectively. Therefore, instead of (8.2.14) we write

$$F(s) = \frac{A}{s} + \frac{B(s + 1) + C}{(s + 1)^2 + 1}. \quad (8.2.15)$$

Finding a common denominator yields

$$F(s) = \frac{A[(s + 1)^2 + 1] + B(s + 1)s + Cs}{s[(s + 1)^2 + 1]}. \quad (8.2.16)$$

If (8.2.13) and (8.2.16) are to be equivalent, then

$$A[(s + 1)^2 + 1] + B(s + 1)s + Cs = 1 - s(5 + 3s).$$

This is true for all s if it's true for three distinct values of s . Choosing $s = 0, -1$, and 1 yields the system

$$\begin{aligned}2A &= 1 \\ A - C &= 3 \\ 5A + 2B + C &= -7.\end{aligned}$$

Solving this system yields

$$A = \frac{1}{2}, \quad B = -\frac{7}{2}, \quad C = -\frac{5}{2}.$$

Hence, from (8.2.15),

$$F(s) = \frac{1}{2s} - \frac{7}{2} \frac{s + 1}{(s + 1)^2 + 1} - \frac{5}{2} \frac{1}{(s + 1)^2 + 1}.$$

Therefore

$$\begin{aligned}\mathcal{L}^{-1}(F) &= \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s}\right) - \frac{7}{2}\mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+1}\right) - \frac{5}{2}\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2+1}\right) \\ &= \frac{1}{2} - \frac{7}{2}e^{-t}\cos t - \frac{5}{2}e^{-t}\sin t.\end{aligned}$$

Example 8.2.9 Find the inverse Laplace transform of

$$F(s) = \frac{8+3s}{(s^2+1)(s^2+4)}. \quad (8.2.17)$$

Solution The form for the partial fraction expansion is

$$F(s) = \frac{A+Bs}{s^2+1} + \frac{C+Ds}{s^2+4}.$$

The coefficients A , B , C and D can be obtained by finding a common denominator and equating the resulting numerator to the numerator in (8.2.17). However, since there's no first power of s in the denominator of (8.2.17), there's an easier way: the expansion of

$$F_1(s) = \frac{1}{(s^2+1)(s^2+4)}$$

can be obtained quickly by using Heaviside's method to expand

$$\frac{1}{(x+1)(x+4)} = \frac{1}{3}\left(\frac{1}{x+1} - \frac{1}{x+4}\right)$$

and then setting $x = s^2$ to obtain

$$\frac{1}{(s^2+1)(s^2+4)} = \frac{1}{3}\left(\frac{1}{s^2+1} - \frac{1}{s^2+4}\right).$$

Multiplying this by $8+3s$ yields

$$F(s) = \frac{8+3s}{(s^2+1)(s^2+4)} = \frac{1}{3}\left(\frac{8+3s}{s^2+1} - \frac{8+3s}{s^2+4}\right).$$

Therefore

$$\mathcal{L}^{-1}(F) = \frac{8}{3}\sin t + \cos t - \frac{4}{3}\sin 2t - \cos 2t.$$

Section 8.3

Example 8.3.1 In Example 8.1.4 we saw that

$$\mathcal{L}(\cos \omega t) = \frac{s}{s^2 + \omega^2}.$$

Applying (8.3.1) with $f(t) = \cos \omega t$ shows that

$$\mathcal{L}(-\omega \sin \omega t) = s \frac{s}{s^2 + \omega^2} - 1 = -\frac{\omega^2}{s^2 + \omega^2}.$$

Therefore

$$\mathcal{L}(\sin \omega t) = \frac{\omega}{s^2 + \omega^2},$$

which agrees with the corresponding result obtained in 8.1.4. ■

Example 8.3.2 Use the Laplace transform to solve the initial value problem

$$y'' - 6y' + 5y = 3e^{2t}, \quad y(0) = 2, \quad y'(0) = 3. \quad (8.3.6)$$

Solution Taking Laplace transforms of both sides of the differential equation in (8.3.6) yields

$$\mathcal{L}(y'' - 6y' + 5y) = \mathcal{L}(3e^{2t}) = \frac{3}{s-2},$$

which we rewrite as

$$\mathcal{L}(y'') - 6\mathcal{L}(y') + 5\mathcal{L}(y) = \frac{3}{s-2}. \quad (8.3.7)$$

Now denote $\mathcal{L}(y) = Y(s)$. Theorem 8.3.2 and the initial conditions in (8.3.6) imply that

$$\mathcal{L}(y') = sY(s) - y(0) = sY(s) - 2$$

and

$$\mathcal{L}(y'') = s^2Y(s) - sy'(0) - sy(0) = s^2Y(s) - 3 - 2s.$$

Substituting from the last two equations into (8.3.7) yields

$$(s^2Y(s) - 3 - 2s) - 6(sY(s) - 2) + 5Y(s) = \frac{3}{s-2}.$$

Therefore

$$(s^2 - 6s + 5)Y(s) = \frac{3}{s-2} + (3 + 2s) + 6(-2), \quad (8.3.8)$$

so

$$(s-5)(s-1)Y(s) = \frac{3 + (s-2)(2s-9)}{s-2},$$

and

$$Y(s) = \frac{3 + (s-2)(2s-9)}{(s-2)(s-5)(s-1)}.$$

Heaviside's method yields the partial fraction expansion

$$Y(s) = -\frac{1}{s-2} + \frac{1}{2} \frac{1}{s-5} + \frac{5}{2} \frac{1}{s-1},$$

and taking the inverse transform of this yields

$$y = -e^{2t} + \frac{1}{2}e^{5t} + \frac{5}{2}e^t$$

as the solution of (8.3.6). ■

Example 8.3.3 Use the Laplace transform to solve the initial value problem

$$2y'' + 3y' + y = 8e^{-2t}, \quad y(0) = -4, \quad y'(0) = 2. \quad (8.3.14)$$

Solution The characteristic polynomial is

$$p(s) = 2s^2 + 3s + 1 = (2s+1)(s+1)$$

and

$$F(s) = \mathcal{L}(8e^{-2t}) = \frac{8}{s+2},$$

so (8.3.13) becomes

$$(2s+1)(s+1)Y(s) = \frac{8}{s+2} + 2(2-4s) + 3(-4).$$

Solving for $Y(s)$ yields

$$Y(s) = \frac{4(1-(s+2)(s+1))}{(s+1/2)(s+1)(s+2)}.$$

Heaviside's method yields the partial fraction expansion

$$Y(s) = \frac{4}{3} \frac{1}{s+1/2} - \frac{8}{s+1} + \frac{8}{3} \frac{1}{s+2},$$

so the solution of (8.3.14) is

$$y = \mathcal{L}^{-1}(Y(s)) = \frac{4}{3}e^{-t/2} - 8e^{-t} + \frac{8}{3}e^{-2t}$$

(Figure 8.3.1).

Example 8.3.4 Solve the initial value problem

$$y'' + 2y' + 2y = 1, \quad y(0) = -3, \quad y'(0) = 1. \quad (8.3.15)$$

Solution The characteristic polynomial is

$$p(s) = s^2 + 2s + 2 = (s+1)^2 + 1$$

and

$$F(s) = \mathcal{L}(1) = \frac{1}{s},$$

so (8.3.13) becomes

$$[(s+1)^2 + 1]Y(s) = \frac{1}{s} + 1 \cdot (1-3s) + 2(-3).$$

Solving for $Y(s)$ yields

$$Y(s) = \frac{1-s(5+3s)}{s[(s+1)^2 + 1]}.$$

In Example 8.2.8 we found the inverse transform of this function to be

$$y = \frac{1}{2} - \frac{7}{2}e^{-t} \cos t - \frac{5}{2}e^{-t} \sin t$$

(Figure 8.3.2), which is therefore the solution of (8.3.15).

Section 8.4

Example 8.4.1 Use the table of Laplace transforms to find the Laplace transform of

$$f(t) = \begin{cases} 2t + 1, & 0 \leq t < 2, \\ 3t, & t \geq 2 \end{cases} \quad (8.4.1)$$

(Figure 8.4.1).

Solution Since the formula for f changes at $t = 2$, we write

$$\begin{aligned} \mathcal{L}(f) &= \int_0^{\infty} e^{-st} f(t) dt \\ &= \int_0^2 e^{-st} (2t + 1) dt + \int_2^{\infty} e^{-st} (3t) dt. \end{aligned} \quad (8.4.2)$$

To relate the first term to a Laplace transform, we add and subtract

$$\int_2^{\infty} e^{-st} (2t + 1) dt$$

in (8.4.2) to obtain

$$\begin{aligned} \mathcal{L}(f) &= \int_0^{\infty} e^{-st} (2t + 1) dt + \int_2^{\infty} e^{-st} (3t - 2t - 1) dt \\ &= \int_0^{\infty} e^{-st} (2t + 1) dt + \int_2^{\infty} e^{-st} (t - 1) dt \\ &= \mathcal{L}(2t + 1) + \int_2^{\infty} e^{-st} (t - 1) dt. \end{aligned} \quad (8.4.3)$$

To relate the last integral to a Laplace transform, we make the change of variable $x = t - 2$ and rewrite the integral as

$$\begin{aligned} \int_2^{\infty} e^{-st} (t - 1) dt &= \int_0^{\infty} e^{-s(x+2)} (x + 1) dx \\ &= e^{-2s} \int_0^{\infty} e^{-sx} (x + 1) dx. \end{aligned}$$

Since the symbol used for the variable of integration has no effect on the value of a definite integral, we can now replace x by the more standard t and write

$$\int_2^{\infty} e^{-st} (t - 1) dt = e^{-2s} \int_0^{\infty} e^{-st} (t + 1) dt = e^{-2s} \mathcal{L}(t + 1).$$

This and (8.4.3) imply that

$$\mathcal{L}(f) = \mathcal{L}(2t + 1) + e^{-2s} \mathcal{L}(t + 1).$$

Now we can use the table of Laplace transforms to find that

$$\mathcal{L}(f) = \frac{2}{s^2} + \frac{1}{s} + e^{-2s} \left(\frac{1}{s^2} + \frac{1}{s} \right). \quad \blacksquare$$

Example 8.4.2 Find

$$\mathcal{L}\left(u(t-1)(t^2+1)\right).$$

Solution Here $\tau = 1$ and $g(t) = t^2 + 1$, so

$$g(t+1) = (t+1)^2 + 1 = t^2 + 2t + 2.$$

Since

$$\mathcal{L}(g(t+1)) = \frac{2}{s^3} + \frac{2}{s^2} + \frac{2}{s},$$

Theorem 8.4.1 implies that

$$\mathcal{L}\left(u(t-1)(t^2+1)\right) = e^{-s} \left(\frac{2}{s^3} + \frac{2}{s^2} + \frac{2}{s} \right).$$

Example 8.4.3 Use Theorem 8.4.1 to find the Laplace transform of the function

$$f(t) = \begin{cases} 2t+1, & 0 \leq t < 2, \\ 3t, & t \geq 2, \end{cases}$$

from Example 8.4.1.

Solution We first write f in the form (8.4.6) as

$$f(t) = 2t+1 + u(t-2)(t-1).$$

Therefore

$$\begin{aligned} \mathcal{L}(f) &= \mathcal{L}(2t+1) + \mathcal{L}(u(t-2)(t-1)) \\ &= \mathcal{L}(2t+1) + e^{-2s} \mathcal{L}(t+1) \quad (\text{from Theorem 8.4.1}) \\ &= \frac{2}{s^2} + \frac{1}{s} + e^{-2s} \left(\frac{1}{s^2} + \frac{1}{s} \right), \end{aligned}$$

which is the result obtained in Example 8.4.1. ■

Example 8.4.4 Find the Laplace transform of

$$f(t) = \begin{cases} 1, & 0 \leq t < 2, \\ -2t+1, & 2 \leq t < 3, \\ 3t, & 3 \leq t < 5, \\ t-1, & t \geq 5 \end{cases} \quad (8.4.7)$$

(Figure 8.4.3).

Solution In terms of step functions,

$$\begin{aligned} f(t) &= 1 + u(t-2)(-2t+1-1) + u(t-3)(3t+2t-1) \\ &\quad + u(t-5)(t-1-3t), \end{aligned}$$

or

$$f(t) = 1 - 2u(t-2)t + u(t-3)(5t-1) - u(t-5)(2t+1).$$

Now Theorem 8.4.1 implies that

$$\begin{aligned}\mathcal{L}(f) &= \mathcal{L}(1) - 2e^{-2s}\mathcal{L}(t+2) + e^{-3s}\mathcal{L}(5(t+3)-1) - e^{-5s}\mathcal{L}(2(t+5)+1) \\ &= \mathcal{L}(1) - 2e^{-2s}\mathcal{L}(t+2) + e^{-3s}\mathcal{L}(5t+14) - e^{-5s}\mathcal{L}(2t+11) \\ &= \frac{1}{s} - 2e^{-2s}\left(\frac{1}{s^2} + \frac{2}{s}\right) + e^{-3s}\left(\frac{5}{s^2} + \frac{14}{s}\right) - e^{-5s}\left(\frac{2}{s^2} + \frac{11}{s}\right). \blacksquare\end{aligned}$$

Example 8.4.5 Find the Laplace transform of

$$f(t) = \begin{cases} \sin t, & 0 \leq t < \frac{\pi}{2}, \\ \cos t - 3 \sin t, & \frac{\pi}{2} \leq t < \pi, \\ 3 \cos t, & t \geq \pi \end{cases} \quad (8.4.10)$$

(Figure 8.4.4).

Solution In terms of step functions,

$$f(t) = \sin t + u(t - \pi/2)(\cos t - 4 \sin t) + u(t - \pi)(2 \cos t + 3 \sin t).$$

Now Theorem 8.4.1 implies that

$$\begin{aligned}\mathcal{L}(f) &= \mathcal{L}(\sin t) + e^{-\frac{\pi}{2}s}\mathcal{L}(\cos(t + \frac{\pi}{2}) - 4 \sin(t + \frac{\pi}{2})) \\ &\quad + e^{-\pi s}\mathcal{L}(2 \cos(t + \pi) + 3 \sin(t + \pi)).\end{aligned} \quad (8.4.11)$$

Since

$$\cos\left(t + \frac{\pi}{2}\right) - 4 \sin\left(t + \frac{\pi}{2}\right) = -\sin t - 4 \cos t$$

and

$$2 \cos(t + \pi) + 3 \sin(t + \pi) = -2 \cos t - 3 \sin t,$$

we see from (8.4.11) that

$$\begin{aligned}\mathcal{L}(f) &= \mathcal{L}(\sin t) - e^{-\pi s/2}\mathcal{L}(\sin t + 4 \cos t) - e^{-\pi s}\mathcal{L}(2 \cos t + 3 \sin t) \\ &= \frac{1}{s^2 + 1} - e^{-\frac{\pi}{2}s}\left(\frac{1 + 4s}{s^2 + 1}\right) - e^{-\pi s}\left(\frac{3 + 2s}{s^2 + 1}\right). \blacksquare\end{aligned}$$

Example 8.4.6 Use (8.4.12) to find

$$\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^2}\right).$$

Solution To apply (8.4.12) we let $\tau = 2$ and $G(s) = 1/s^2$. Then $g(t) = t$ and (8.4.12) implies that

$$\mathcal{L}^{-1}\left(\frac{e^{-2s}}{s^2}\right) = u(t-2)(t-2). \blacksquare$$

Example 8.4.7 Find the inverse Laplace transform h of

$$H(s) = \frac{1}{s^2} - e^{-s} \left(\frac{1}{s^2} + \frac{2}{s} \right) + e^{-4s} \left(\frac{4}{s^3} + \frac{1}{s} \right),$$

and find distinct formulas for h on appropriate intervals.

Solution Let

$$G_0(s) = \frac{1}{s^2}, \quad G_1(s) = \frac{1}{s^2} + \frac{2}{s}, \quad G_2(s) = \frac{4}{s^3} + \frac{1}{s}.$$

Then

$$g_0(t) = t, \quad g_1(t) = t + 2, \quad g_2(t) = 2t^2 + 1.$$

Hence, (8.4.12) and the linearity of $\mathcal{L}^{-1}$ imply that

$$\begin{aligned} h(t) &= \mathcal{L}^{-1}(G_0(s)) - \mathcal{L}^{-1}(e^{-s}G_1(s)) + \mathcal{L}^{-1}(e^{-4s}G_2(s)) \\ &= t - u(t-1)[(t-1)+2] + u(t-4)[2(t-4)^2+1] \\ &= t - u(t-1)(t+1) + u(t-4)(2t^2-16t+33), \end{aligned}$$

which can also be written as

$$h(t) = \begin{cases} t, & 0 \leq t < 1, \\ -1, & 1 \leq t < 4, \\ 2t^2 - 16t + 32, & t \geq 4. \end{cases} \quad \blacksquare$$

Example 8.4.8 Find the inverse transform of

$$H(s) = \frac{2s}{s^2+4} - e^{-\frac{\pi}{2}s} \frac{3s+1}{s^2+9} + e^{-\pi s} \frac{s+1}{s^2+6s+10}.$$

Solution Let

$$G_0(s) = \frac{2s}{s^2+4}, \quad G_1(s) = -\frac{(3s+1)}{s^2+9},$$

and

$$G_2(s) = \frac{s+1}{s^2+6s+10} = \frac{(s+3)-2}{(s+3)^2+1}.$$

Then

$$g_0(t) = 2 \cos 2t, \quad g_1(t) = -3 \cos 3t - \frac{1}{3} \sin 3t,$$

and

$$g_2(t) = e^{-3t}(\cos t - 2 \sin t).$$

Therefore (8.4.12) and the linearity of $\mathcal{L}^{-1}$ imply that

$$\begin{aligned} h(t) &= 2 \cos 2t - u(t-\pi/2) \left[3 \cos 3(t-\pi/2) + \frac{1}{3} \sin 3 \left(t - \frac{\pi}{2} \right) \right] \\ &\quad + u(t-\pi) e^{-3(t-\pi)} [\cos(t-\pi) - 2 \sin(t-\pi)]. \end{aligned}$$

Using the trigonometric identities (8.4.8) and (8.4.9), we can rewrite this as

$$\begin{aligned} h(t) = & 2 \cos 2t + u(t - \pi/2) \left(3 \sin 3t - \frac{1}{3} \cos 3t \right) \\ & - u(t - \pi) e^{-3(t-\pi)} (\cos t - 2 \sin t) \end{aligned} \quad (8.4.13)$$

(Figure 8.4.5).

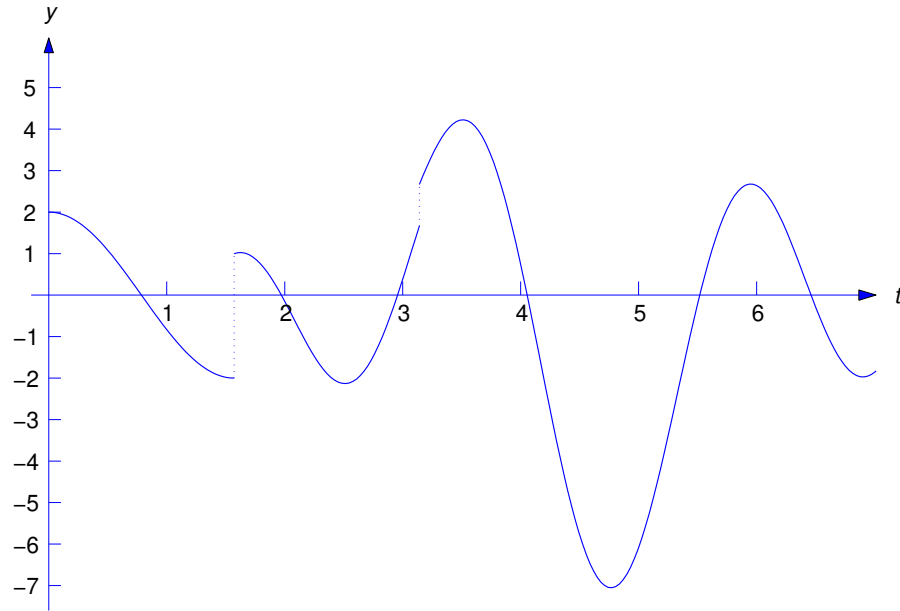


Figure 8.4.5 The piecewise continuous function (8.4.13)

Section 8.5

Example 8.5.1 Solve the initial value problem

$$y'' + y = f(t), \quad y(0) = 2, \quad y'(0) = -1, \quad (8.5.3)$$

where

$$f(t) = \begin{cases} 1, & 0 \leq t < \frac{\pi}{2}, \\ -1, & t \geq \frac{\pi}{2}. \end{cases}$$

Solution The initial value problem in Step 1 is

$$y'' + y = 1, \quad y(0) = 2, \quad y'(0) = -1.$$

We leave it to you to verify that its solution is

$$y_0 = 1 + \cos t - \sin t.$$

Doing Step 2 yields $y_0(\pi/2) = 0$ and $y'_0(\pi/2) = -1$, so the second initial value problem is

$$y'' + y = -1, \quad y\left(\frac{\pi}{2}\right) = 0, \quad y'\left(\frac{\pi}{2}\right) = -1.$$

We leave it to you to verify that the solution of this problem is

$$y_1 = -1 + \cos t + \sin t.$$

Hence, the solution of (8.5.3) is

$$y = \begin{cases} 1 + \cos t - \sin t, & 0 \leq t < \frac{\pi}{2}, \\ -1 + \cos t + \sin t, & t \geq \frac{\pi}{2} \end{cases} \quad (8.5.4)$$

(Figure 8.5.1).

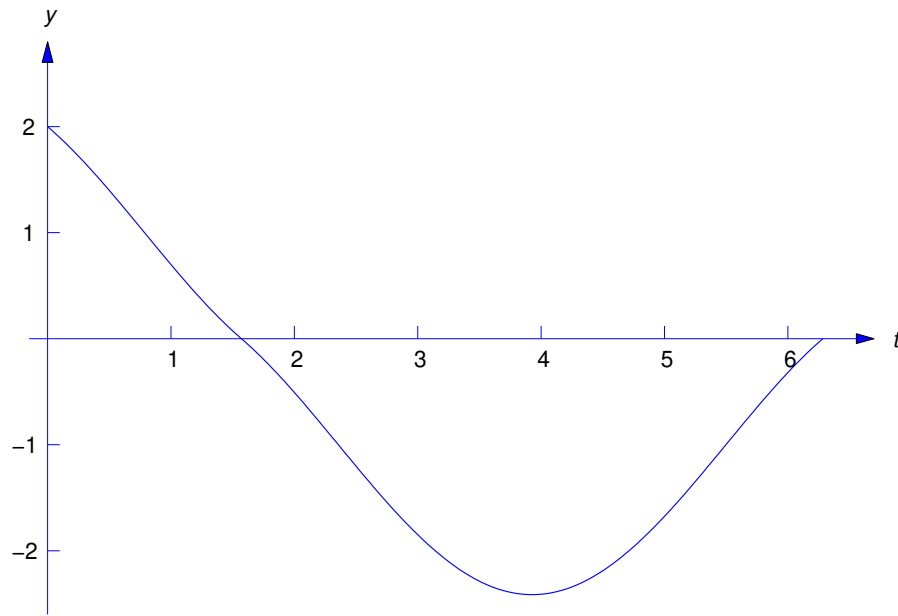


Figure 8.5.1 Graph of (8.5.4)

Example 8.5.2 Use the Laplace transform to solve the initial value problem

$$y'' + y = f(t), \quad y(0) = 2, \quad y'(0) = -1, \quad (8.5.5)$$

where

$$f(t) = \begin{cases} 1, & 0 \leq t < \frac{\pi}{2}, \\ -1, & t \geq \frac{\pi}{2}. \end{cases}$$

Solution Here

$$f(t) = 1 - 2u\left(t - \frac{\pi}{2}\right),$$

so Theorem 8.4.1 (with $g(t) = 1$) implies that

$$\mathcal{L}(f) = \frac{1 - 2e^{-\pi s/2}}{s}.$$

Therefore, transforming (8.5.5) yields

$$(s^2 + 1)Y(s) = \frac{1 - 2e^{-\pi s/2}}{s} - 1 + 2s,$$

so

$$Y(s) = (1 - 2e^{-\pi s/2})G(s) + \frac{2s - 1}{s^2 + 1}, \quad (8.5.6)$$

with

$$G(s) = \frac{1}{s(s^2 + 1)}.$$

The form for the partial fraction expansion of G is

$$\frac{1}{s(s^2 + 1)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 1}. \quad (8.5.7)$$

Multiplying through by $s(s^2 + 1)$ yields

$$A(s^2 + 1) + (Bs + C)s = 1,$$

or

$$(A + B)s^2 + Cs + A = 1.$$

Equating coefficients of like powers of s on the two sides of this equation shows that $A = 1$, $B = -A = -1$ and $C = 0$. Hence, from (8.5.7),

$$G(s) = \frac{1}{s} - \frac{s}{s^2 + 1}.$$

Therefore

$$g(t) = 1 - \cos t.$$

From this, (8.5.6), and Theorem 8.4.2,

$$y = 1 - \cos t - 2u\left(t - \frac{\pi}{2}\right)\left(1 - \cos\left(t - \frac{\pi}{2}\right)\right) + 2\cos t - \sin t.$$

Simplifying this (recalling that $\cos(t - \pi/2) = \sin t$) yields

$$y = 1 + \cos t - \sin t - 2u\left(t - \frac{\pi}{2}\right)(1 - \sin t),$$

or

$$y = \begin{cases} 1 + \cos t - \sin t, & 0 \leq t < \frac{\pi}{2}, \\ -1 + \cos t + \sin t, & t \geq \frac{\pi}{2}, \end{cases}$$

which is the result obtained in Example 8.5.1.

Example 8.5.3 Solve the initial value problem

$$y'' - y = f(t), \quad y(0) = -1, \quad y'(0) = 2, \quad (8.5.8)$$

where

$$f(t) = \begin{cases} t, & 0 \leq t < 1, \\ 1, & t \geq 1. \end{cases}$$

Solution Here

$$f(t) = t - u(t-1)(t-1),$$

so

$$\begin{aligned}\mathcal{L}(f) &= \mathcal{L}(t) - \mathcal{L}(u(t-1)(t-1)) \\ &= \mathcal{L}(t) - e^{-s}\mathcal{L}(t) \text{ (from Theorem 8.4.1)} \\ &= \frac{1}{s^2} - \frac{e^{-s}}{s^2}.\end{aligned}$$

Since transforming (8.5.8) yields

$$(s^2 - 1)Y(s) = \mathcal{L}(f) + 2 - s,$$

we see that

$$Y(s) = (1 - e^{-s})H(s) + \frac{2-s}{s^2-1}, \quad (8.5.9)$$

where

$$H(s) = \frac{1}{s^2(s^2-1)} = \frac{1}{s^2-1} - \frac{1}{s^2};$$

therefore

$$h(t) = \sinh t - t. \quad (8.5.10)$$

Since

$$\mathcal{L}^{-1}\left(\frac{2-s}{s^2-1}\right) = 2 \sinh t - \cosh t,$$

we conclude from (8.5.9), (8.5.10), and Theorem 8.4.1 that

$$y = \sinh t - t - u(t-1)(\sinh(t-1) - t + 1) + 2 \sinh t - \cosh t,$$

or

$$y = 3 \sinh t - \cosh t - t - u(t-1)(\sinh(t-1) - t + 1) \quad (8.5.11)$$

We leave it to you to verify that y and y' are continuous and y'' has limits from the right and left at $t_1 = 1$.

Example 8.5.4 Solve the initial value problem

$$y'' + y = f(t), \quad y(0) = 0, \quad y'(0) = 0, \quad (8.5.12)$$

where

$$f(t) = \begin{cases} 0, & 0 \leq t < \frac{\pi}{4}, \\ \cos 2t, & \frac{\pi}{4} \leq t < \pi, \\ 0, & t \geq \pi. \end{cases}$$

Solution Here

$$f(t) = u(t - \pi/4) \cos 2t - u(t - \pi) \cos 2t,$$

so

$$\begin{aligned}
 \mathcal{L}(f) &= \mathcal{L}(u(t - \pi/4) \cos 2t) - \mathcal{L}(u(t - \pi) \cos 2t) \\
 &= e^{-\pi s/4} \mathcal{L}(\cos 2(t + \pi/4)) - e^{-\pi s} \mathcal{L}(\cos 2(t + \pi)) \\
 &= -e^{-\pi s/4} \mathcal{L}(\sin 2t) - e^{-\pi s} \mathcal{L}(\cos 2t) \\
 &= -\frac{2e^{-\pi s/4}}{s^2 + 4} - \frac{se^{-\pi s}}{s^2 + 4}.
 \end{aligned}$$

Since transforming (8.5.12) yields

$$(s^2 + 1)Y(s) = \mathcal{L}(f),$$

we see that

$$Y(s) = e^{-\pi s/4} H_1(s) + e^{-\pi s} H_2(s), \quad (8.5.13)$$

where

$$H_1(s) = -\frac{2}{(s^2 + 1)(s^2 + 4)} \quad \text{and} \quad H_2(s) = -\frac{s}{(s^2 + 1)(s^2 + 4)}. \quad (8.5.14)$$

To simplify the required partial fraction expansions, we first write

$$\frac{1}{(x + 1)(x + 4)} = \frac{1}{3} \left[\frac{1}{x + 1} - \frac{1}{x + 4} \right].$$

Setting $x = s^2$ and substituting the result in (8.5.14) yields

$$H_1(s) = -\frac{2}{3} \left[\frac{1}{s^2 + 1} - \frac{1}{s^2 + 4} \right] \quad \text{and} \quad H_2(s) = -\frac{1}{3} \left[\frac{s}{s^2 + 1} - \frac{s}{s^2 + 4} \right].$$

The inverse transforms are

$$h_1(t) = -\frac{2}{3} \sin t + \frac{1}{3} \sin 2t \quad \text{and} \quad h_2(t) = -\frac{1}{3} \cos t + \frac{1}{3} \cos 2t.$$

From (8.5.13) and Theorem 8.4.2,

$$y = u\left(t - \frac{\pi}{4}\right) h_1\left(t - \frac{\pi}{4}\right) + u(t - \pi) h_2(t - \pi). \quad (8.5.15)$$

Since

$$\begin{aligned}
 h_1\left(t - \frac{\pi}{4}\right) &= -\frac{2}{3} \sin\left(t - \frac{\pi}{4}\right) + \frac{1}{3} \sin 2\left(t - \frac{\pi}{4}\right) \\
 &= -\frac{\sqrt{2}}{3} (\sin t - \cos t) - \frac{1}{3} \cos 2t
 \end{aligned}$$

and

$$\begin{aligned}
 h_2(t - \pi) &= -\frac{1}{3} \cos(t - \pi) + \frac{1}{3} \cos 2(t - \pi) \\
 &= \frac{1}{3} \cos t + \frac{1}{3} \cos 2t,
 \end{aligned}$$

(8.5.15) can be rewritten as

$$y = -\frac{1}{3}u\left(t - \frac{\pi}{4}\right)\left(\sqrt{2}(\sin t - \cos t) + \cos 2t\right) + \frac{1}{3}u(t - \pi)(\cos t + \cos 2t)$$

or

$$y = \begin{cases} 0, & 0 \leq t < \frac{\pi}{4}, \\ -\frac{\sqrt{2}}{3}(\sin t - \cos t) - \frac{1}{3}\cos 2t, & \frac{\pi}{4} \leq t < \pi, \\ -\frac{\sqrt{2}}{3}\sin t + \frac{1 + \sqrt{2}}{3}\cos t, & t \geq \pi. \end{cases} \quad (8.5.16)$$

We leave it to you to verify that y and y' are continuous and y'' has limits from the right and left at $t_1 = \pi/4$ and $t_2 = \pi$ (Figure 8.5.2).

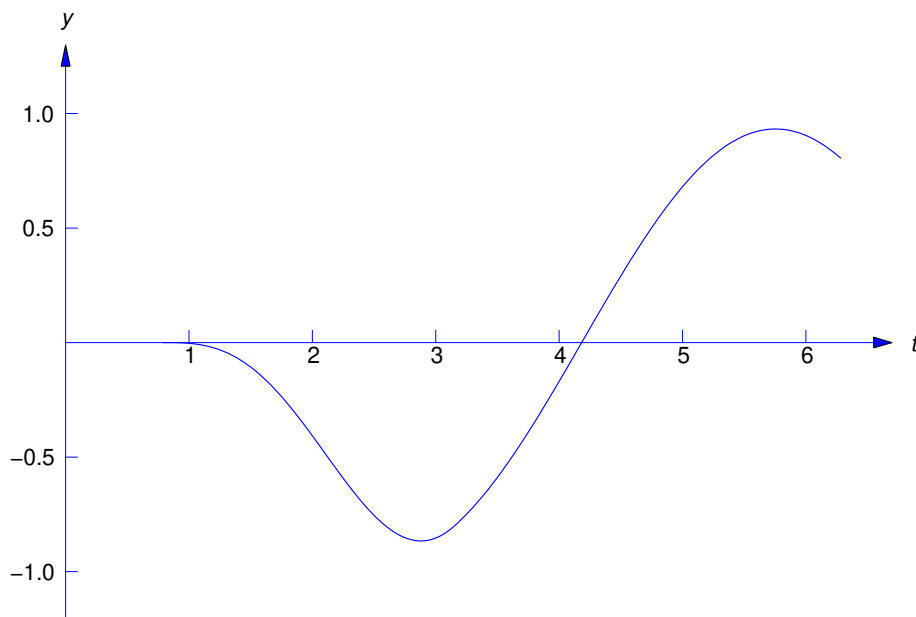


Figure 8.5.2 Graph of (8.5.16)

Section 8.6

Example 8.6.1 Let

$$f(t) = e^{at} \quad \text{and} \quad g(t) = e^{bt} \quad (a \neq b).$$

Verify that $\mathcal{L}(f * g) = \mathcal{L}(f)\mathcal{L}(g)$, as implied by the convolution theorem.

Solution We first compute

$$\begin{aligned} (f * g)(t) &= \int_0^t e^{a\tau} e^{b(t-\tau)} d\tau = e^{bt} \int_0^t e^{(a-b)\tau} d\tau \\ &= e^{bt} \frac{e^{(a-b)\tau}}{a-b} \Big|_0^t = \frac{e^{bt} [e^{(a-b)t} - 1]}{a-b} \\ &= \frac{e^{at} - e^{bt}}{a-b}. \end{aligned}$$

Since

$$e^{at} \leftrightarrow \frac{1}{s-a} \quad \text{and} \quad e^{bt} \leftrightarrow \frac{1}{s-b},$$

it follows that

$$\begin{aligned} \mathcal{L}(f * g) &= \frac{1}{a-b} \left[\frac{1}{s-a} - \frac{1}{s-b} \right] \\ &= \frac{1}{(s-a)(s-b)} \\ &= \mathcal{L}(e^{at})\mathcal{L}(e^{bt}) = \mathcal{L}(f)\mathcal{L}(g). \end{aligned}$$

Example 8.6.2 Find a formula for the solution of the initial value problem

$$y'' - 2y' + y = f(t), \quad y(0) = k_0, \quad y'(0) = k_1. \quad (8.6.7)$$

Solution Taking Laplace transforms in (8.6.7) yields

$$(s^2 - 2s + 1)Y(s) = F(s) + (k_1 + k_0s) - 2k_0.$$

Therefore

$$\begin{aligned} Y(s) &= \frac{1}{(s-1)^2}F(s) + \frac{k_1 + k_0s - 2k_0}{(s-1)^2} \\ &= \frac{1}{(s-1)^2}F(s) + \frac{k_0}{s-1} + \frac{k_1 - k_0}{(s-1)^2}. \end{aligned}$$

From the table of Laplace transforms,

$$\mathcal{L}^{-1} \left(\frac{k_0}{s-1} + \frac{k_1 - k_0}{(s-1)^2} \right) = e^t (k_0 + (k_1 - k_0)t).$$

Since

$$\frac{1}{(s-1)^2} \leftrightarrow te^t \quad \text{and} \quad F(s) \leftrightarrow f(t),$$

the convolution theorem implies that

$$\mathcal{L}^{-1} \left(\frac{1}{(s-1)^2} F(s) \right) = \int_0^t \tau e^\tau f(t-\tau) d\tau.$$

Therefore the solution of (8.6.7) is

$$y(t) = e^t (k_0 + (k_1 - k_0)t) + \int_0^t \tau e^\tau f(t-\tau) d\tau.$$

Example 8.6.3 Find a formula for the solution of the initial value problem

$$y'' + 4y = f(t), \quad y(0) = k_0, \quad y'(0) = k_1. \quad (8.6.8)$$

Solution Taking Laplace transforms in (8.6.8) yields

$$(s^2 + 4)Y(s) = F(s) + k_1 + k_0s.$$

Therefore

$$Y(s) = \frac{1}{(s^2 + 4)}F(s) + \frac{k_1 + k_0s}{s^2 + 4}.$$

From the table of Laplace transforms,

$$\mathcal{L}^{-1}\left(\frac{k_1 + k_0s}{s^2 + 4}\right) = k_0 \cos 2t + \frac{k_1}{2} \sin 2t.$$

Since

$$\frac{1}{(s^2 + 4)} \leftrightarrow \frac{1}{2} \sin 2t \quad \text{and} \quad F(s) \leftrightarrow f(t),$$

the convolution theorem implies that

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2 + 4)}F(s)\right) = \frac{1}{2} \int_0^t f(t - \tau) \sin 2\tau \, d\tau.$$

Therefore the solution of (8.6.8) is

$$y(t) = k_0 \cos 2t + \frac{k_1}{2} \sin 2t + \frac{1}{2} \int_0^t f(t - \tau) \sin 2\tau \, d\tau.$$

Example 8.6.4 Find a formula for the solution of the initial value problem

$$y'' + 2y' + 2y = f(t), \quad y(0) = k_0, \quad y'(0) = k_1. \quad (8.6.9)$$

Solution Taking Laplace transforms in (8.6.9) yields

$$(s^2 + 2s + 2)Y(s) = F(s) + k_1 + k_0s + 2k_0.$$

Therefore

$$\begin{aligned} Y(s) &= \frac{1}{(s+1)^2 + 1} F(s) + \frac{k_1 + k_0s + 2k_0}{(s+1)^2 + 1} \\ &= \frac{1}{(s+1)^2 + 1} F(s) + \frac{(k_1 + k_0) + k_0(s+1)}{(s+1)^2 + 1}. \end{aligned}$$

From the table of Laplace transforms,

$$\mathcal{L}^{-1}\left(\frac{(k_1 + k_0) + k_0(s+1)}{(s+1)^2 + 1}\right) = e^{-t} ((k_1 + k_0) \sin t + k_0 \cos t).$$

Since

$$\frac{1}{(s+1)^2 + 1} \leftrightarrow e^{-t} \sin t \quad \text{and} \quad F(s) \leftrightarrow f(t),$$

the convolution theorem implies that

$$\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2 + 1} F(s)\right) = \int_0^t f(t - \tau) e^{-\tau} \sin \tau \, d\tau.$$

Therefore the solution of (8.6.9) is

$$y(t) = e^{-t} ((k_1 + k_0) \sin t + k_0 \cos t) + \int_0^t f(t - \tau) e^{-\tau} \sin \tau \, d\tau. \quad (8.6.10)$$

Example 8.6.5 Evaluate the convolution integral

$$h(t) = \int_0^t (t - \tau)^5 \tau^7 d\tau.$$

Solution We could evaluate this integral by expanding $(t - \tau)^5$ in powers of τ and then integrating. However, the convolution theorem provides an easier way. The integral is the convolution of $f(t) = t^5$ and $g(t) = t^7$. Since

$$t^5 \leftrightarrow \frac{5!}{s^6} \quad \text{and} \quad t^7 \leftrightarrow \frac{7!}{s^8},$$

the convolution theorem implies that

$$h(t) \leftrightarrow \frac{5!7!}{s^{14}} = \frac{5!7!}{13!} \frac{13!}{s^{14}},$$

where we have written the second equality because

$$\frac{13!}{s^{14}} \leftrightarrow t^{13}.$$

Hence,

$$h(t) = \frac{5!7!}{13!} t^{13}.$$

Example 8.6.6 Use the convolution theorem and a partial fraction expansion to evaluate the convolution integral

$$h(t) = \int_0^t \sin a(t - \tau) \cos b\tau d\tau \quad (|a| \neq |b|).$$

Solution Since

$$\sin at \leftrightarrow \frac{a}{s^2 + a^2} \quad \text{and} \quad \cos bt \leftrightarrow \frac{s}{s^2 + b^2},$$

the convolution theorem implies that

$$H(s) = \frac{a}{s^2 + a^2} \frac{s}{s^2 + b^2}.$$

Expanding this in a partial fraction expansion yields

$$H(s) = \frac{a}{b^2 - a^2} \left[\frac{s}{s^2 + a^2} - \frac{s}{s^2 + b^2} \right].$$

Therefore

$$h(t) = \frac{a}{b^2 - a^2} (\cos at - \cos bt).$$

Example 8.6.7 Solve the integral equation

$$y(t) = 1 + 2 \int_0^t e^{-2(t-\tau)} y(\tau) d\tau. \quad (8.6.12)$$

Solution Taking Laplace transforms in (8.6.12) yields

$$Y(s) = \frac{1}{s} + \frac{2}{s+2} Y(s),$$

and solving this for $Y(s)$ yields

$$Y(s) = \frac{1}{s} + \frac{2}{s^2}.$$

Hence,

$$y(t) = 1 + 2t.$$

Section 8.7

Example 8.7.1 Find the solution of the initial value problem

$$y'' - 2y' + y = \delta(t - t_0), \quad y(0) = 0, \quad y'(0) = 0, \quad (8.7.15)$$

where $t_0 > 0$. Then interpret the solution for the case where $t_0 = 0$.

Solution Here

$$w = \mathcal{L}^{-1} \left(\frac{1}{s^2 - 2s + 1} \right) = \mathcal{L}^{-1} \left(\frac{1}{(s - 1)^2} \right) = te^t,$$

so Definition 8.7.2 yields

$$y = u(t - t_0)(t - t_0)e^{t-t_0}$$

as the solution of (8.7.15) if $t_0 > 0$. If $t_0 = 0$, then (8.7.15) doesn't have a solution; however, $y = u(t)te^t$ (which we would usually write simply as $y = te^t$) is the solution of the modified initial value problem

$$y'' - 2y' + y = \delta(t), \quad y(0) = 0, \quad y'_-(0) = 0.$$

The graph of $y = u(t - t_0)(t - t_0)e^{t-t_0}$ is shown in Figure 8.7.3

■

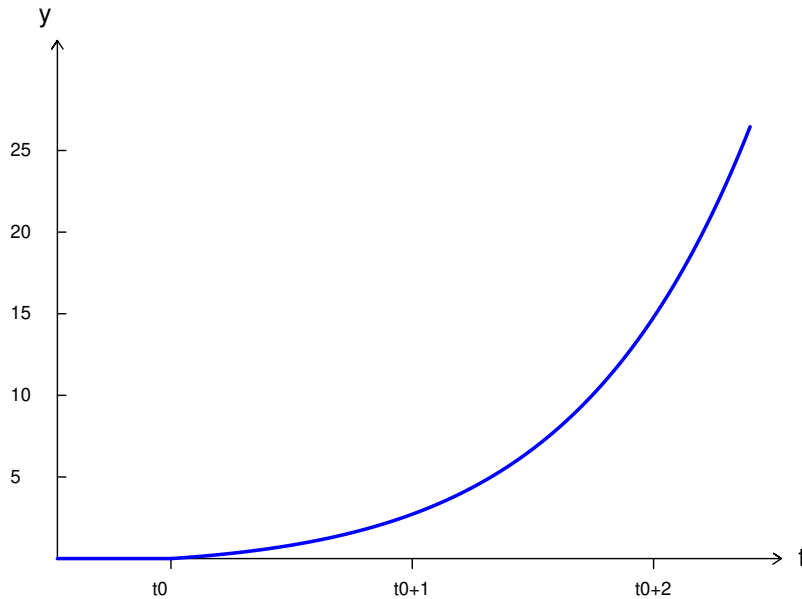


Figure 8.7.3 $y = u(t - t_0)(t - t_0)e^{t-t_0}$

Example 8.7.2 Solve the initial value problem

$$y'' + 6y' + 5y = 3e^{-2t} + 2\delta(t - 1), \quad y(0) = -3, \quad y'(0) = 2. \quad (8.7.16)$$

Solution We leave it to you to show that the solution of

$$y'' + 6y' + 5y = 3e^{-2t}, \quad y(0) = -3, \quad y'(0) = 2$$

is

$$\hat{y} = -e^{-2t} + \frac{1}{2}e^{-5t} - \frac{5}{2}e^{-t}.$$

Since

$$\begin{aligned} w(t) &= \mathcal{L}^{-1}\left(\frac{1}{s^2 + 6s + 5}\right) = \mathcal{L}^{-1}\left(\frac{1}{(s+1)(s+5)}\right) \\ &= \frac{1}{4}\mathcal{L}^{-1}\left(\frac{1}{s+1} - \frac{1}{s+5}\right) = \frac{e^{-t} - e^{-5t}}{4}, \end{aligned}$$

the solution of (8.7.16) is

$$y = -e^{-2t} + \frac{1}{2}e^{-5t} - \frac{5}{2}e^{-t} + u(t-1)\frac{e^{-(t-1)} - e^{-5(t-1)}}{2} \quad (8.7.17)$$

(Figure 8.7.4)

■.

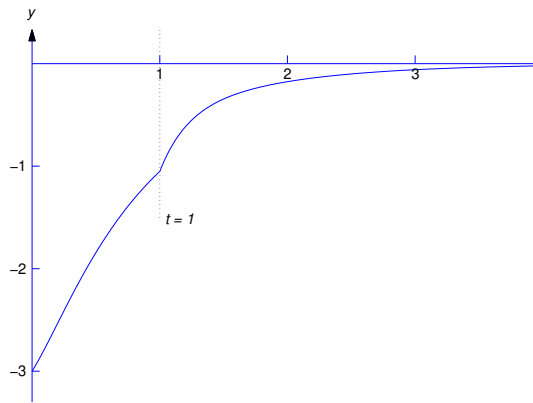


Figure 8.7.4 Graph of (8.7.17)

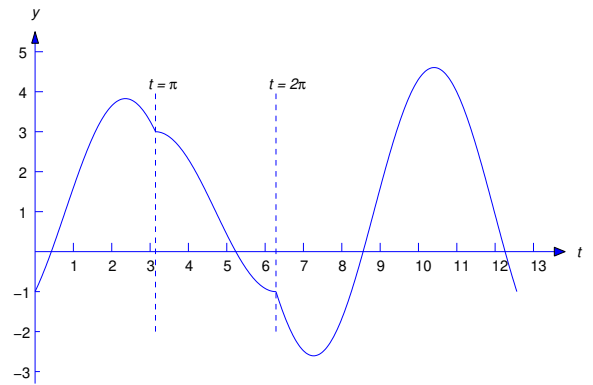


Figure 8.7.5 Graph of (8.7.19)

Example 8.7.3 Solve the initial value problem

$$y'' + y = 1 + 2\delta(t - \pi) - 3\delta(t - 2\pi), \quad y(0) = -1, \quad y'(0) = 2. \quad (8.7.18)$$

Solution We leave it to you to show that

$$\hat{y} = 1 - 2\cos t + 2\sin t$$

is the solution of

$$y'' + y = 1, \quad y(0) = -1, \quad y'(0) = 2.$$

Since

$$w = \mathcal{L}^{-1}\left(\frac{1}{s^2 + 1}\right) = \sin t,$$

the solution of (8.7.18) is

$$\begin{aligned} y &= 1 - 2\cos t + 2\sin t + 2u(t - \pi)\sin(t - \pi) - 3u(t - 2\pi)\sin(t - 2\pi) \\ &= 1 - 2\cos t + 2\sin t - 2u(t - \pi)\sin t - 3u(t - 2\pi)\sin t, \end{aligned}$$

or

$$y = \begin{cases} 1 - 2\cos t + 2\sin t, & 0 \leq t < \pi, \\ 1 - 2\cos t, & \pi \leq t < 2\pi, \\ 1 - 2\cos t - 3\sin t, & t \geq 2\pi \end{cases} \quad (8.7.19)$$

(Figure 8.7.5).

Section 9.1

Example 9.1.1 The equation

$$x^3 y''' - x^2 y'' - 2xy' + 6y = 0 \quad (9.1.5)$$

is normal and has the solutions $y_1 = x^2$, $y_2 = x^3$, and $y_3 = 1/x$ on $(-\infty, 0)$ and $(0, \infty)$. Show that $\{y_1, y_2, y_3\}$ is linearly independent on $(-\infty, 0)$ and $(0, \infty)$. Then find the general solution of (9.1.5) on $(-\infty, 0)$ and $(0, \infty)$.

Solution Suppose

$$c_1 x^2 + c_2 x^3 + \frac{c_3}{x} = 0 \quad (9.1.6)$$

on $(0, \infty)$. We must show that $c_1 = c_2 = c_3 = 0$. Differentiating (9.1.6) twice yields the system

$$\begin{aligned} c_1 x^2 + c_2 x^3 + \frac{c_3}{x} &= 0 \\ 2c_1 x + 3c_2 x^2 - \frac{c_3}{x^2} &= 0 \\ 2c_1 + 6c_2 x + \frac{2c_3}{x^3} &= 0. \end{aligned} \quad (9.1.7)$$

If (9.1.7) holds for all x in $(0, \infty)$, then it certainly holds at $x = 1$; therefore,

$$\begin{aligned} c_1 + c_2 + c_3 &= 0 \\ 2c_1 + 3c_2 - c_3 &= 0 \\ 2c_1 + 6c_2 + 2c_3 &= 0. \end{aligned} \quad (9.1.8)$$

By solving this system directly, you can verify that it has only the trivial solution $c_1 = c_2 = c_3 = 0$; however, for our purposes it's more useful to recall from linear algebra that a homogeneous linear system of n equations in n unknowns has only the trivial solution if its determinant is nonzero. Since the determinant of (9.1.8) is

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & -1 \\ 2 & 6 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & -3 \\ 2 & 4 & 0 \end{vmatrix} = 12,$$

it follows that (9.1.8) has only the trivial solution, so $\{y_1, y_2, y_3\}$ is linearly independent on $(0, \infty)$. Now Theorem 9.1.2 implies that

$$y = c_1 x^2 + c_2 x^3 + \frac{c_3}{x}$$

is the general solution of (9.1.5) on $(0, \infty)$. To see that this is also true on $(-\infty, 0)$, assume that (9.1.6) holds on $(-\infty, 0)$. Setting $x = -1$ in (9.1.7) yields

$$\begin{aligned} c_1 - c_2 - c_3 &= 0 \\ -2c_1 + 3c_2 - c_3 &= 0 \\ 2c_1 - 6c_2 - 2c_3 &= 0. \end{aligned}$$

Since the determinant of this system is

$$\begin{vmatrix} 1 & -1 & -1 \\ -2 & 3 & -1 \\ 2 & -6 & -2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ -2 & 1 & -3 \\ 2 & -4 & 0 \end{vmatrix} = -12,$$

it follows that $c_1 = c_2 = c_3 = 0$; that is, $\{y_1, y_2, y_3\}$ is linearly independent on $(-\infty, 0)$.

Example 9.1.2 The equation

$$y^{(4)} + y''' - 7y'' - y' + 6y = 0 \quad (9.1.9)$$

is normal and has the solutions $y_1 = e^x$, $y_2 = e^{-x}$, $y_3 = e^{2x}$, and $y_4 = e^{-3x}$ on $(-\infty, \infty)$. (Verify.) Show that $\{y_1, y_2, y_3, y_4\}$ is linearly independent on $(-\infty, \infty)$. Then find the general solution of (9.1.9).

Solution Suppose c_1, c_2, c_3 , and c_4 are constants such that

$$c_1 e^x + c_2 e^{-x} + c_3 e^{2x} + c_4 e^{-3x} = 0 \quad (9.1.10)$$

for all x . We must show that $c_1 = c_2 = c_3 = c_4 = 0$. Differentiating (9.1.10) three times yields the system

$$\begin{aligned} c_1 e^x + c_2 e^{-x} + c_3 e^{2x} + c_4 e^{-3x} &= 0 \\ c_1 e^x - c_2 e^{-x} + 2c_3 e^{2x} - 3c_4 e^{-3x} &= 0 \\ c_1 e^x + c_2 e^{-x} + 4c_3 e^{2x} + 9c_4 e^{-3x} &= 0 \\ c_1 e^x - c_2 e^{-x} + 8c_3 e^{2x} - 27c_4 e^{-3x} &= 0. \end{aligned} \quad (9.1.11)$$

If (9.1.11) holds for all x , then it certainly holds for $x = 0$. Therefore

$$\begin{aligned} c_1 + c_2 + c_3 + c_4 &= 0 \\ c_1 - c_2 + 2c_3 - 3c_4 &= 0 \\ c_1 + c_2 + 4c_3 + 9c_4 &= 0 \\ c_1 - c_2 + 8c_3 - 27c_4 &= 0. \end{aligned}$$

The determinant of this system is

$$\begin{aligned} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & -3 \\ 1 & 1 & 4 & 9 \\ 1 & -1 & 8 & -27 \end{vmatrix} &= \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 1 & -4 \\ 0 & 0 & 3 & 8 \\ 0 & -2 & 7 & -28 \end{vmatrix} = \begin{vmatrix} -2 & 1 & -4 \\ 0 & 3 & 8 \\ -2 & 7 & -28 \end{vmatrix} \\ &= \begin{vmatrix} -2 & 1 & -4 \\ 0 & 3 & 8 \\ 0 & 6 & -24 \end{vmatrix} = -2 \begin{vmatrix} 3 & 8 \\ 6 & -24 \end{vmatrix} = 240, \end{aligned} \quad (9.1.12)$$

so the system has only the trivial solution $c_1 = c_2 = c_3 = c_4 = 0$. Now Theorem 9.1.2 implies that

$$y = c_1 e^x + c_2 e^{-x} + c_3 e^{2x} + c_4 e^{-3x}$$

is the general solution of (9.1.9).

Example 9.1.3 In Example 9.1.1 we saw that the solutions $y_1 = x^2$, $y_2 = x^3$, and $y_3 = 1/x$ of

$$x^3 y''' - x^2 y'' - 2xy' + 6y = 0$$

are linearly independent on $(-\infty, 0)$ and $(0, \infty)$. Calculate the Wronskian of $\{y_1, y_2, y_3\}$.

Solution If $x \neq 0$, then

$$W(x) = \begin{vmatrix} x^2 & x^3 & \frac{1}{x} \\ 2x & 3x^2 & -\frac{1}{x^2} \\ 2 & 6x & \frac{2}{x^3} \end{vmatrix} = 2x^3 \begin{vmatrix} 1 & x & \frac{1}{x^3} \\ 2 & 3x & -\frac{1}{x^3} \\ 1 & 3x & \frac{1}{x^3} \end{vmatrix},$$

where we factored x^2 , x , and 2 out of the first, second, and third rows of $W(x)$, respectively. Adding the second row of the last determinant to the first and third rows yields

$$W(x) = 2x^3 \begin{vmatrix} 3 & 4x & 0 \\ 2 & 3x & -\frac{1}{x^3} \\ 3 & 6x & 0 \end{vmatrix} = 2x^3 \left(\frac{1}{x^3} \right) \begin{vmatrix} 3 & 4x \\ 3 & 6x \end{vmatrix} = 12x.$$

Therefore $W(x) \neq 0$ on $(-\infty, 0)$ and $(0, \infty)$.

Example 9.1.4 In Example 9.1.2 we saw that the solutions $y_1 = e^x$, $y_2 = e^{-x}$, $y_3 = e^{2x}$, and $y_4 = e^{-3x}$ of

$$y^{(4)} + y''' - 7y'' - y' + 6y = 0$$

are linearly independent on every open interval. Calculate the Wronskian of $\{y_1, y_2, y_3, y_4\}$.

Solution For all x ,

$$W(x) = \begin{vmatrix} e^x & e^{-x} & e^{2x} & e^{-3x} \\ e^x & -e^{-x} & 2e^{2x} & -3e^{-3x} \\ e^x & e^{-x} & 4e^{2x} & 9e^{-3x} \\ e^x & -e^{-x} & 8e^{2x} & -27e^{-3x} \end{vmatrix}.$$

Factoring the exponential common factor from each row yields

$$W(x) = e^{-x} \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 2 & -3 \\ 1 & 1 & 4 & 9 \\ 1 & -1 & 8 & -27 \end{vmatrix} = 240e^{-x},$$

from (9.1.12).

Section 9.2

Example 9.2.1

(a) Find the general solution of

$$y''' - 6y'' + 11y' - 6y = 0. \quad (9.2.3)$$

(b) Solve the initial value problem

$$y''' - 6y'' + 11y' - 6y = 0, \quad y(0) = 4, \quad y'(0) = 5, \quad y''(0) = 9. \quad (9.2.4)$$

Solution The characteristic polynomial of (9.2.3) is

$$p(r) = r^3 - 6r^2 + 11r - 6 = (r - 1)(r - 2)(r - 3).$$

Therefore $\{e^x, e^{2x}, e^{3x}\}$ is a set of solutions of (9.2.3). It is a fundamental set, since its Wronskian is

$$W(x) = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = e^{6x} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = 2e^{6x} \neq 0.$$

Therefore the general solution of (9.2.3) is

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}. \quad (9.2.5)$$

Solution (b) We must determine c_1, c_2 and c_3 in (9.2.5) so that y satisfies the initial conditions in (9.2.4). Differentiating (9.2.5) twice yields

$$\begin{aligned} y' &= c_1 e^x + 2c_2 e^{2x} + 3c_3 e^{3x} \\ y'' &= c_1 e^x + 4c_2 e^{2x} + 9c_3 e^{3x}. \end{aligned} \quad (9.2.6)$$

Setting $x = 0$ in (9.2.5) and (9.2.6) and imposing the initial conditions yields

$$\begin{aligned} c_1 + c_2 + c_3 &= 4 \\ c_1 + 2c_2 + 3c_3 &= 5 \\ c_1 + 4c_2 + 9c_3 &= 9. \end{aligned}$$

The solution of this system is $c_1 = 4, c_2 = -1, c_3 = 1$. Therefore the solution of (9.2.4) is

$$y = 4e^x - e^{2x} + e^{3x}$$

(Figure 9.2.1). ■

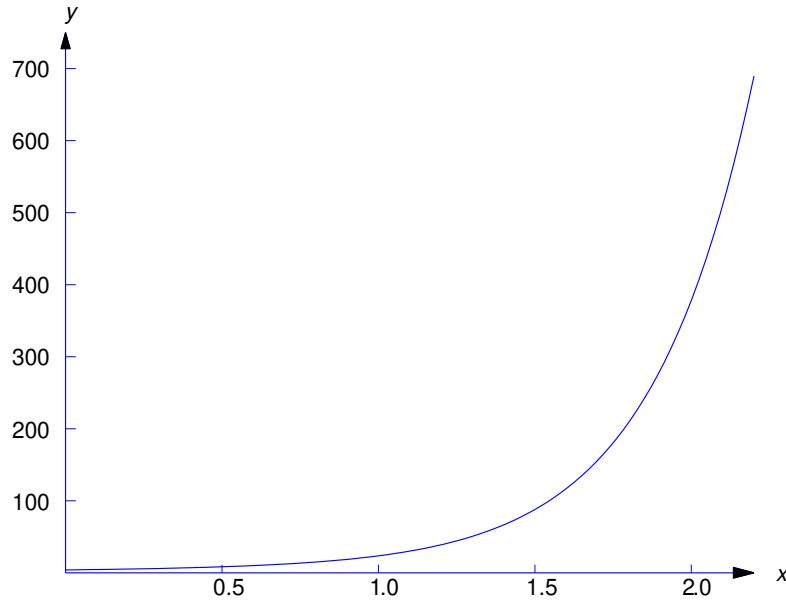


Figure 9.2.1 $y = 4e^x - e^{2x} + e^{3x}$

Example 9.2.2 Consider the polynomial

$$p(r) = r^3 - r^2 + r - 1$$

and the associated polynomial operator

$$p(D) = D^3 - D^2 + D - 1.$$

Since $p(r)$ can be factored as

$$p(r) = (r - 1)(r^2 + 1) = (r^2 + 1)(r - 1),$$

it's reasonable to expect that $p(D)$ can be factored as

$$p(D) = (D - 1)(D^2 + 1) = (D^2 + 1)(D - 1). \quad (9.2.7)$$

However, before we can make this assertion we must *define* what we mean by saying that two operators are equal, and what we mean by the products of operators in (9.2.7). We say that two operators are equal if they apply to the same functions and always produce the same result. The definition of the products in (9.2.7) is this: if y is any three-times differentiable function then

- (a) $(D - 1)(D^2 + 1)y$ is the function obtained by first applying $D^2 + 1$ to y and then applying $D - 1$ to the resulting function
- (b) $(D^2 + 1)(D - 1)y$ is the function obtained by first applying $D - 1$ to y and then applying $D^2 + 1$ to the resulting function.

From (a),

$$\begin{aligned} (D - 1)(D^2 + 1)y &= (D - 1)[(D^2 + 1)y] \\ &= (D - 1)(y'' + y) = D(y'' + y) - (y'' + y) \\ &= (y''' + y') - (y'' + y) \\ &= y''' - y'' + y' - y = (D^3 - D^2 + D - 1)y. \end{aligned} \quad (9.2.8)$$

This implies that

$$(D - 1)(D^2 + 1) = (D^3 - D^2 + D - 1).$$

From (b),

$$\begin{aligned} (D^2 + 1)(D - 1)y &= (D^2 + 1)[(D - 1)y] \\ &= (D^2 + 1)(y' - y) = D^2(y' - y) + (y' - y) \\ &= (y''' - y'') + (y' - y) \\ &= y''' - y'' + y' - y = (D^3 - D^2 + D - 1)y, \end{aligned} \quad (9.2.9)$$

$$(D^2 + 1)(D - 1) = (D^3 - D^2 + D - 1),$$

which completes the justification of (9.2.7).

Example 9.2.3 Use the result of Example 9.2.2 to find the general solution of

$$y''' - y'' + y' - y = 0. \quad (9.2.10)$$

Solution From (9.2.8), we can rewrite (9.2.10) as

$$(D - 1)(D^2 + 1)y = 0,$$

which implies that any solution of $(D^2 + 1)y = 0$ is a solution of (9.2.10). Therefore $y_1 = \cos x$ and $y_2 = \sin x$ are solutions of (9.2.10).

From (9.2.9), we can rewrite (9.2.10) as

$$(D^2 + 1)(D - 1)y = 0,$$

which implies that any solution of $(D - 1)y = 0$ is a solution of (9.2.10). Therefore $y_3 = e^x$ is a solution of (9.2.10).

The Wronskian of $\{\cos x, \sin x, e^x\}$ is

$$W(x) = \begin{vmatrix} \cos x & \sin x & e^x \\ -\sin x & \cos x & e^x \\ -\cos x & -\sin x & e^x \end{vmatrix}.$$

Since

$$W(0) = \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ -1 & 0 & 1 \end{vmatrix} = 2,$$

$\{\cos x, \sin x, e^x\}$ is linearly independent and

$$y = c_1 \cos x + c_2 \sin x + c_3 e^x$$

is the general solution of (9.2.10).

Example 9.2.4 Find the general solution of

$$y^{(4)} - 16y = 0. \quad (9.2.11)$$

Solution The characteristic polynomial of (9.2.11) is

$$p(r) = r^4 - 16 = (r^2 - 4)(r^2 + 4) = (r - 2)(r + 2)(r^2 + 4).$$

By arguments similar to those used in Examples 9.2.2 and 9.2.3, it can be shown that (9.2.11) can be written as

$$(D^2 + 4)(D + 2)(D - 2)y = 0$$

or

$$(D^2 + 4)(D - 2)(D + 2)y = 0$$

or

$$(D - 2)(D + 2)(D^2 + 4)y = 0.$$

Therefore y is a solution of (9.2.11) if it's a solution of any of the three equations

$$(D - 2)y = 0, \quad (D + 2)y = 0, \quad (D^2 + 4)y = 0.$$

Hence, $\{e^{2x}, e^{-2x}, \cos 2x, \sin 2x\}$ is a set of solutions of (9.2.11). The Wronskian of this set is

$$W(x) = \begin{vmatrix} e^{2x} & e^{-2x} & \cos 2x & \sin 2x \\ 2e^{2x} & -2e^{-2x} & -2\sin 2x & 2\cos 2x \\ 4e^{2x} & 4e^{-2x} & -4\cos 2x & -4\sin 2x \\ 8e^{2x} & -8e^{-2x} & 8\sin 2x & -8\cos 2x \end{vmatrix}.$$

Since

$$W(0) = \begin{vmatrix} 1 & 1 & 1 & 0 \\ 2 & -2 & 0 & 2 \\ 4 & 4 & -4 & 0 \\ 8 & -8 & 0 & -8 \end{vmatrix} = -512,$$

$\{e^{2x}, e^{-2x}, \cos 2x, \sin 2x\}$ is linearly independent, and

$$y = c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos 2x + c_4 \sin 2x$$

is the general solution of (9.2.11).

Example 9.2.5 Find the general solution of

$$y''' + 3y'' + 3y' + y = 0. \quad (9.2.18)$$

Solution The characteristic polynomial of (9.2.18) is

$$p(r) = r^3 + 3r^2 + 3r + 1 = (r + 1)^3.$$

Therefore (9.2.18) can be written as

$$(D + 1)^3 y = 0,$$

so Theorem 9.2.1 implies that the general solution of (9.2.18) is

$$y = e^{-x}(c_1 + c_2 x + c_3 x^2). \quad \blacksquare$$

Example 9.2.6 Find the general solution of

$$(D^2 + 4D + 13)^3 y = 0. \quad (9.2.19)$$

Solution The characteristic polynomial of (9.2.19) is

$$p(r) = (r^2 + 4r + 13)^3 = \left((r + 2)^2 + 9\right)^3.$$

Therefore (9.2.19) can be written as

$$[(D + 2)^2 + 9]^3 y = 0,$$

so Theorem 9.2.2 implies that the general solution of (9.2.19) is

$$y = (a_1 + a_2 x + a_3 x^2) e^{-2x} \cos 3x + (b_1 + b_2 x + b_3 x^2) e^{-2x} \sin 3x.$$

Example 9.2.7 Find the general solution of

$$y^{(4)} + 4y''' + 6y'' + 4y' = 0. \quad (9.2.20)$$

Solution The characteristic polynomial of (9.2.20) is

$$\begin{aligned} p(r) &= r^4 + 4r^3 + 6r^2 + 4r \\ &= r(r^3 + 4r^2 + 6r + 4) \\ &= r(r+2)(r^2 + 2r + 2) \\ &= r(r+2)[(r+1)^2 + 1]. \end{aligned}$$

Therefore (9.2.20) can be written as

$$[(D+1)^2 + 1](D+2)Dy = 0.$$

Fundamental sets of solutions of

$$[(D+1)^2 + 1]y = 0, \quad (D+2)y = 0, \quad \text{and} \quad Dy = 0.$$

are given by

$$\{e^{-x} \cos x, e^{-x} \sin x\}, \quad \{e^{-2x}\}, \quad \text{and} \quad \{1\},$$

respectively. Therefore the general solution of (9.2.20) is

$$y = e^{-x}(c_1 \cos x + c_2 \sin x) + c_3 e^{-2x} + c_4.$$

Example 9.2.8 Find a fundamental set of solutions of

$$[(D+1)^2 + 1]^2(D-1)^3(D+1)D^2y = 0. \quad (9.2.21)$$

Solution A fundamental set of solutions of (9.2.21) can be obtained by combining fundamental sets of solutions of

$$\begin{aligned} [(D+1)^2 + 1]^2 y &= 0, \quad (D-1)^3 y = 0, \\ (D+1)y &= 0, \quad \text{and} \quad D^2 y = 0. \end{aligned}$$

Fundamental sets of solutions of these equations are given by

$$\begin{aligned} \{e^{-x} \cos x, x e^{-x} \cos x, e^{-x} \sin x, x e^{-x} \sin x\}, \quad \{e^x, x e^x, x^2 e^x\}, \\ \{e^{-x}\}, \quad \text{and} \quad \{1, x\}, \end{aligned}$$

respectively. These ten functions form a fundamental set of solutions of (9.2.21).

Section 9.3

Example 9.3.1 Find a particular solution of

$$y''' + 3y'' + 2y' - y = e^x(21 + 24x + 28x^2 + 5x^3). \quad (9.3.2)$$

Solution Substituting

$$\begin{aligned} y &= u e^x, \\ y' &= e^x(u' + u), \\ y'' &= e^x(u'' + 2u' + u), \\ y''' &= e^x(u''' + 3u'' + 3u' + u) \end{aligned}$$

into (9.3.2) and canceling e^x yields

$$(u''' + 3u'' + 3u' + u) + 3(u'' + 2u' + u) + 2(u' + u) - u = 21 + 24x + 28x^2 + 5x^3,$$

or

$$u''' + 6u'' + 11u' + 5u = 21 + 24x + 28x^2 + 5x^3. \quad (9.3.3)$$

Since the unknown u appears on the left, we can see that (9.3.3) has a particular solution of the form

$$u_p = A + Bx + Cx^2 + Dx^3.$$

Then

$$\begin{aligned} u_p' &= B + 2Cx + 3Dx^2 \\ u_p'' &= 2C + 6Dx \\ u_p''' &= 6D. \end{aligned}$$

Substituting from the last four equations into the left side of (9.3.3) yields

$$\begin{aligned} u_p''' + 6u_p'' + 11u_p' + 5u_p &= 6D + 6(2C + 6Dx) + 11(B + 2Cx + 3Dx^2) \\ &\quad + 5(A + Bx + Cx^2 + Dx^3) \\ &= (5A + 11B + 12C + 6D) + (5B + 22C + 36D)x \\ &\quad + (5C + 33D)x^2 + 5Dx^3. \end{aligned}$$

Comparing coefficients of like powers of x on the right sides of this equation and (9.3.3) shows that u_p satisfies (9.3.3) if

$$\begin{aligned} 5D &= 5 \\ 5C + 33D &= 28 \\ 5B + 22C + 36D &= 24 \\ 5A + 11B + 12C + 6D &= 21. \end{aligned}$$

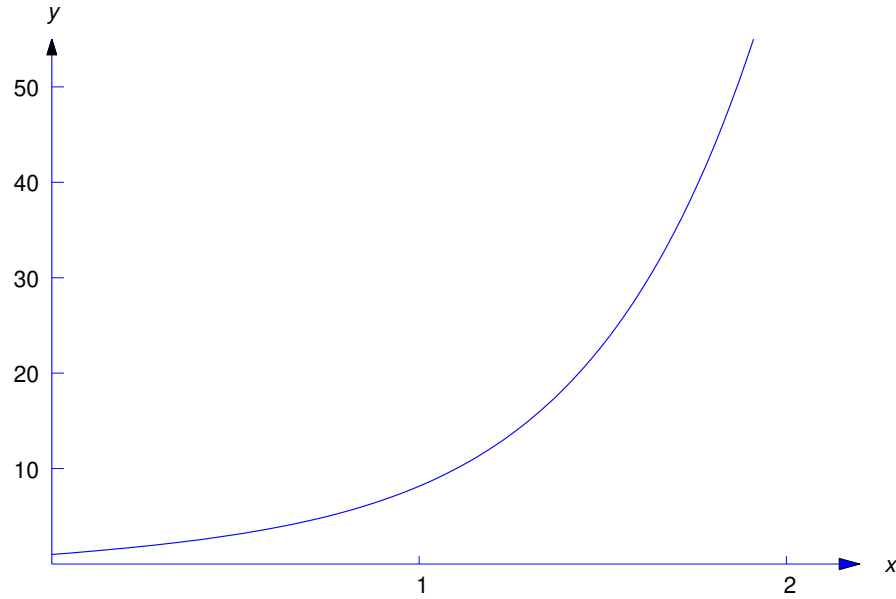
Solving these equations successively yields $D = 1$, $C = -1$, $B = 2$, $A = 1$. Therefore

$$u_p = 1 + 2x - x^2 + x^3$$

is a particular solution of (9.3.3), so

$$y_p = e^x u_p = e^x(1 + 2x - x^2 + x^3)$$

is a particular solution of (9.3.2) (Figure 9.3.1).

Figure 9.3.1 $y_p = e^x(1 + 2x - x^2 + x^3)$

Example 9.3.2 Find a particular solution of

$$y^{(4)} - y''' - 6y'' + 4y' + 8y = e^{2x}(4 + 19x + 6x^2). \quad (9.3.4)$$

Solution Substituting

$$\begin{aligned} y &= ue^{2x}, \\ y' &= e^{2x}(u' + 2u), \\ y'' &= e^{2x}(u'' + 4u' + 4u), \\ y''' &= e^{2x}(u''' + 6u'' + 12u' + 8u), \\ y^{(4)} &= e^{2x}(u^{(4)} + 8u''' + 24u'' + 32u' + 16u) \end{aligned}$$

into (9.3.4) and canceling e^{2x} yields

$$\begin{aligned} (u^{(4)} + 8u''' + 24u'' + 32u' + 16u) - (u''' + 6u'' + 12u' + 8u) \\ - 6(u'' + 4u' + 4u) + 4(u' + 2u) + 8u = 4 + 19x + 6x^2, \end{aligned}$$

or

$$u^{(4)} + 7u''' + 12u'' = 4 + 19x + 6x^2. \quad (9.3.5)$$

Since neither u nor u' appear on the left, we can see that (9.3.5) has a particular solution of the form

$$u_p = Ax^2 + Bx^3 + Cx^4. \quad (9.3.6)$$

Then

$$\begin{aligned} u'_p &= 2Ax + 3Bx^2 + 4Cx^3 \\ u''_p &= 2A + 6Bx + 12Cx^2 \\ u'''_p &= 6B + 24Cx \\ u^{(4)}_p &= 24C. \end{aligned}$$

Substituting u_p'' , u_p''' , and $u_p^{(4)}$ into the left side of (9.3.5) yields

$$\begin{aligned} u_p^{(4)} + 7u_p''' + 12u_p'' &= 24C + 7(6B + 24Cx) + 12(2A + 6Bx + 12Cx^2) \\ &= (24A + 42B + 24C) + (72B + 168C)x + 144Cx^2. \end{aligned}$$

Comparing coefficients of like powers of x on the right sides of this equation and (9.3.5) shows that u_p satisfies (9.3.5) if

$$\begin{aligned} 144C &= 6 \\ 72B + 168C &= 19 \\ 24A + 42B + 24C &= 4. \end{aligned}$$

Solving these equations successively yields $C = 1/24$, $B = 1/6$, $A = -1/6$. Substituting these into (9.3.6) shows that

$$u_p = \frac{x^2}{24}(-4 + 4x + x^2)$$

is a particular solution of (9.3.5), so

$$y_p = e^{2x}u_p = \frac{x^2 e^{2x}}{24}(-4 + 4x + x^2)$$

is a particular solution of (9.3.4). (Figure 9.3.2).

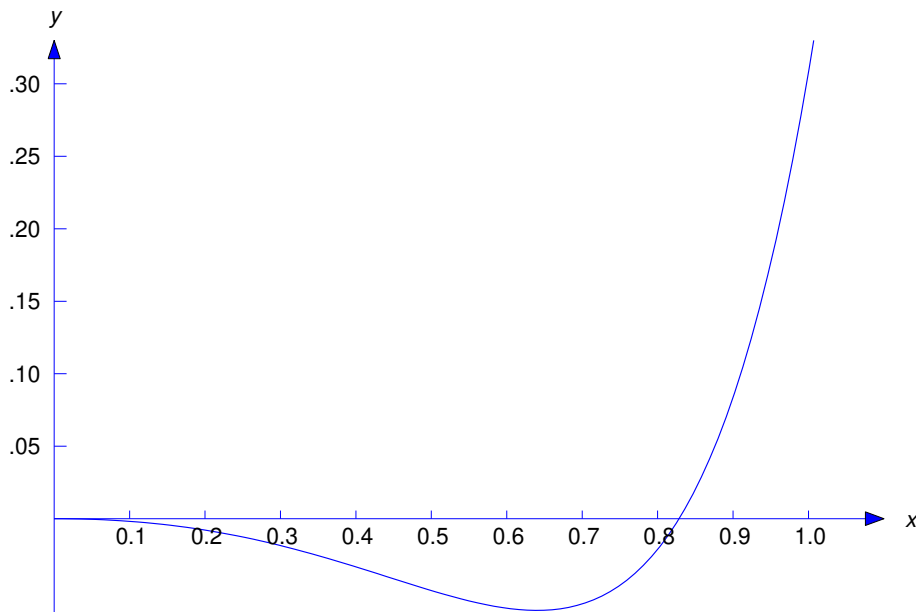


Figure 9.3.2 $y_p = \frac{x^2 e^{2x}}{24}(-4 + 4x + x^2)$

Example 9.3.3 Find a particular solution of

$$y''' + y'' - 4y' - 4y = e^x[(5 - 5x) \cos x + (2 + 5x) \sin x]. \quad (9.3.7)$$

Solution Substituting

$$\begin{aligned} y &= ue^x, \\ y' &= e^x(u' + u), \\ y'' &= e^x(u'' + 2u' + u), \\ y''' &= e^x(u''' + 3u'' + 3u' + u) \end{aligned}$$

into (9.3.7) and canceling e^x yields

$$(u''' + 3u'' + 3u' + u) + (u'' + 2u' + u) - 4(u' + u) - 4u = (5 - 5x) \cos x + (2 + 5x) \sin x,$$

or

$$u''' + 4u'' + u' - 6u = (5 - 5x) \cos x + (2 + 5x) \sin x. \quad (9.3.8)$$

Since $\cos x$ and $\sin x$ are not solutions of the complementary equation

$$u''' + 4u'' + u' - 6u = 0,$$

a theorem analogous to Theorem 5.5.1 implies that (9.3.8) has a particular solution of the form

$$u_p = (A_0 + A_1x) \cos x + (B_0 + B_1x) \sin x. \quad (9.3.9)$$

Then

$$\begin{aligned} u'_p &= (A_1 + B_0 + B_1x) \cos x + (B_1 - A_0 - A_1x) \sin x, \\ u''_p &= (2B_1 - A_0 - A_1x) \cos x - (2A_1 + B_0 + B_1x) \sin x, \\ u'''_p &= -(3A_1 + B_0 + B_1x) \cos x - (3B_1 - A_0 - A_1x) \sin x, \end{aligned}$$

so

$$\begin{aligned} u'''_p + 4u''_p + u'_p - 6u_p &= -[10A_0 + 2A_1 - 8B_1 + 10A_1x] \cos x \\ &\quad - [10B_0 + 2B_1 + 8A_1 + 10B_1x] \sin x. \end{aligned}$$

Comparing the coefficients of $x \cos x$, $x \sin x$, $\cos x$, and $\sin x$ here with the corresponding coefficients in (9.3.8) shows that u_p is a solution of (9.3.8) if

$$\begin{aligned} -10A_1 &= -5 \\ -10B_1 &= 5 \\ -10A_0 - 2A_1 + 8B_1 &= 5 \\ -10B_0 - 2B_1 - 8A_1 &= 2. \end{aligned}$$

Solving the first two equations yields $A_1 = 1/2$, $B_1 = -1/2$. Substituting these into the last two equations yields

$$\begin{aligned} -10A_0 &= 5 + 2A_1 - 8B_1 = 10 \\ -10B_0 &= 2 + 2B_1 + 8A_1 = 5, \end{aligned}$$

so $A_0 = -1$, $B_0 = -1/2$. Substituting $A_0 = -1$, $A_1 = 1/2$, $B_0 = -1/2$, $B_1 = -1/2$ into (9.3.9) shows that

$$u_p = -\frac{1}{2} [(2 - x) \cos x + (1 + x) \sin x]$$

is a particular solution of (9.3.8), so

$$y_p = e^x u_p = -\frac{e^x}{2} [(2 - x) \cos x + (1 + x) \sin x]$$

is a particular solution of (9.3.7) (Figure 9.3.3).

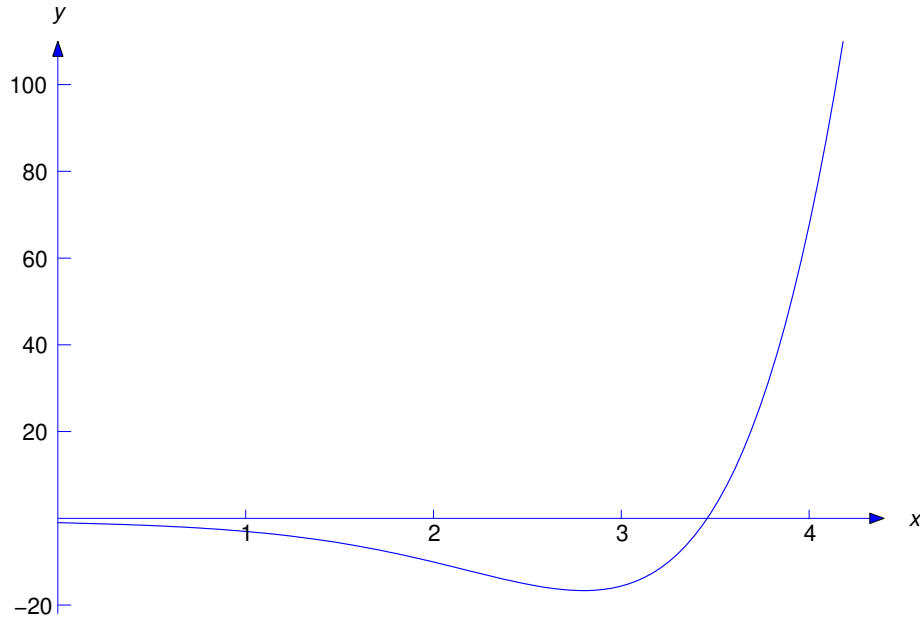


Figure 9.3.3 $y_p = e^x u_p = -\frac{e^x}{2} [(2-x)\cos x + (1+x)\sin x]$

Example 9.3.4 Find a particular solution of

$$y''' + 4y'' + 6y' + 4y = e^{-x} [(1-6x)\cos x - (3+2x)\sin x]. \quad (9.3.10)$$

Solution Substituting

$$\begin{aligned} y &= ue^{-x}, \\ y' &= e^{-x}(u' - u), \\ y'' &= e^{-x}(u'' - 2u' + u), \\ y''' &= e^{-x}(u''' - 3u'' + 3u' - u) \end{aligned}$$

into (9.3.10) and canceling e^{-x} yields

$$(u''' - 3u'' + 3u' - u) + 4(u'' - 2u' + u) + 6(u' - u) + 4u = (1-6x)\cos x - (3+2x)\sin x,$$

or

$$u''' + u'' + u' + u = (1-6x)\cos x - (3+2x)\sin x. \quad (9.3.11)$$

Since $\cos x$ and $\sin x$ are solutions of the complementary equation

$$u''' + u'' + u' + u = 0,$$

a theorem analogous to Theorem 5.5.1 implies that (9.3.11) has a particular solution of the form

$$u_p = (A_0x + A_1x^2)\cos x + (B_0x + B_1x^2)\sin x. \quad (9.3.12)$$

Then

$$\begin{aligned} u'_p &= [A_0 + (2A_1 + B_0)x + B_1x^2]\cos x + [B_0 + (2B_1 - A_0)x - A_1x^2]\sin x, \\ u''_p &= [2A_1 + 2B_0 - (A_0 - 4B_1)x - A_1x^2]\cos x \\ &\quad + [2B_1 - 2A_0 - (B_0 + 4A_1)x - B_1x^2]\sin x, \\ u'''_p &= -[3A_0 - 6B_1 + (6A_1 + B_0)x + B_1x^2]\cos x \\ &\quad - [3B_0 + 6A_1 + (6B_1 - A_0)x - A_1x^2]\sin x, \end{aligned}$$

so

$$\begin{aligned} u_p''' + u_p'' + u_p' + u_p &= -[2A_0 - 2B_0 - 2A_1 - 6B_1 + (4A_1 - 4B_1)x] \cos x \\ &\quad -[2B_0 + 2A_0 - 2B_1 + 6A_1 + (4B_1 + 4A_1)x] \sin x. \end{aligned}$$

Comparing the coefficients of $x \cos x$, $x \sin x$, $\cos x$, and $\sin x$ here with the corresponding coefficients in (9.3.11) shows that u_p is a solution of (9.3.11) if

$$\begin{aligned} -4A_1 + 4B_1 &= -6 \\ -4A_1 - 4B_1 &= -2 \\ -2A_0 + 2B_0 + 2A_1 + 6B_1 &= 1 \\ -2A_0 - 2B_0 - 6A_1 + 2B_1 &= -3. \end{aligned}$$

Solving the first two equations yields $A_1 = 1$, $B_1 = -1/2$. Substituting these into the last two equations yields

$$\begin{aligned} -2A_0 + 2B_0 &= 1 - 2A_1 - 6B_1 = 2 \\ -2A_0 - 2B_0 &= -3 + 6A_1 - 2B_1 = 4, \end{aligned}$$

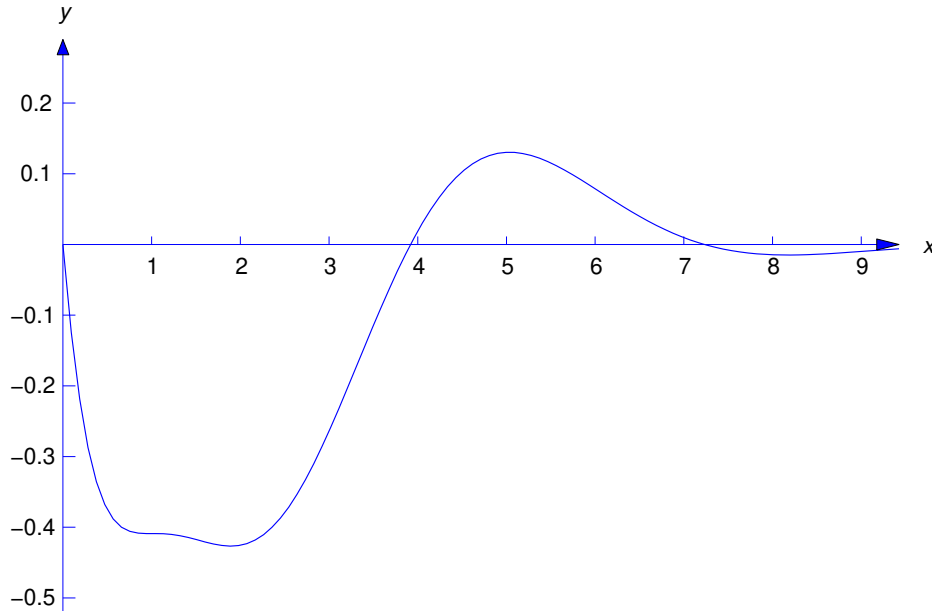


Figure 9.3.4 $y_p = -\frac{x e^{-x}}{2} [(3 - 2x) \cos x + (1 + x) \sin x]$

so $A_0 = -3/2$ and $B_0 = -1/2$. Substituting $A_0 = -3/2$, $A_1 = 1$, $B_0 = -1/2$, $B_1 = -1/2$ into (9.3.12) shows that

$$u_p = -\frac{x}{2} [(3 - 2x) \cos x + (1 + x) \sin x]$$

is a particular solution of (9.3.11), so

$$y_p = e^{-x} u_p = -\frac{x e^{-x}}{2} [(3 - 2x) \cos x + (1 + x) \sin x]$$

(Figure 9.3.4) is a particular solution of (9.3.10).

Section 9.4

Example 9.4.1 Find a particular solution of

$$xy''' - y'' - xy' + y = 8x^2e^x, \quad (9.4.8)$$

given that $y_1 = x$, $y_2 = e^x$, and $y_3 = e^{-x}$ form a fundamental set of solutions of the complementary equation. Then find the general solution of (9.4.8).

Solution We seek a particular solution of (9.4.8) of the form

$$y_p = u_1x + u_2e^x + u_3e^{-x}.$$

The Wronskian of $\{y_1, y_2, y_3\}$ is

$$W(x) = \begin{vmatrix} x & e^x & e^{-x} \\ 1 & e^x & -e^{-x} \\ 0 & e^x & e^{-x} \end{vmatrix},$$

so

$$W_1 = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2,$$

$$W_2 = \begin{vmatrix} x & e^{-x} \\ 1 & -e^{-x} \end{vmatrix} = -e^{-x}(x+1),$$

$$W_3 = \begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix} = e^x(x-1).$$

Expanding W by cofactors of the last row yields

$$W = 0W_1 - e^xW_2 + e^{-x}W_3 = 0(-2) - e^x(-e^{-x}(x+1)) + e^{-x}e^x(x-1) = 2x.$$

Since $F(x) = 8x^2e^x$ and $P_0(x) = x$,

$$\frac{F}{P_0W} = \frac{8x^2e^x}{x \cdot 2x} = 4e^x.$$

Therefore, from (9.4.7)

$$\begin{aligned} u_1' &= 4e^xW_1 = 4e^x(-2) = -8e^x, \\ u_2' &= -4e^xW_2 = -4e^x(-e^{-x}(x+1)) = 4(x+1), \\ u_3' &= 4e^xW_3 = 4e^x(e^x(x-1)) = 4e^{2x}(x-1). \end{aligned}$$

Integrating and taking the constants of integration to be zero yields

$$u_1 = -8e^x, \quad u_2 = 2(x+1)^2, \quad u_3 = e^{2x}(2x-3).$$

Hence,

$$\begin{aligned} y_p &= u_1y_1 + u_2y_2 + u_3y_3 \\ &= (-8e^x)x + e^x(2(x+1)^2) + e^{-x}(e^{2x}(2x-3)) \\ &= e^x(2x^2 - 2x - 1). \end{aligned}$$

Since $-e^x$ is a solution of the complementary equation, we redefine

$$y_p = 2xe^x(x-1).$$

Therefore the general solution of (9.4.8) is

$$y = 2xe^x(x-1) + c_1x + c_2e^x + c_3e^{-x}.$$

Example 9.4.2 Find a particular solution of

$$x^4y^{(4)} + 6x^3y''' + 2x^2y'' - 4xy' + 4y = 12x^2, \quad (9.4.10)$$

given that $y_1 = x$, $y_2 = x^2$, $y_3 = 1/x$ and $y_4 = 1/x^2$ form a fundamental set of solutions of the complementary equation. Then find the general solution of (9.4.10) on $(-\infty, 0)$ and $(0, \infty)$.

Solution We seek a particular solution of (9.4.10) of the form

$$y_p = u_1x + u_2x^2 + \frac{u_3}{x} + \frac{u_4}{x^2}.$$

The Wronskian of $\{y_1, y_2, y_3, y_4\}$ is

$$W(x) = \begin{vmatrix} x & x^2 & 1/x & 1/x^2 \\ 1 & 2x & -1/x^2 & -2/x^3 \\ 0 & 2 & 2/x^3 & 6/x^4 \\ 0 & 0 & -6/x^4 & -24/x^5 \end{vmatrix},$$

so

$$W_1 = \begin{vmatrix} x^2 & 1/x & 1/x^2 \\ 2x & -1/x^2 & -2/x^3 \\ 2 & 2/x^3 & 6/x^4 \end{vmatrix} = -\frac{12}{x^4},$$

$$W_2 = \begin{vmatrix} x & 1/x & 1/x^2 \\ 1 & -1/x^2 & -2/x^3 \\ 0 & 2/x^3 & 6/x^4 \end{vmatrix} = -\frac{6}{x^5},$$

$$W_3 = \begin{vmatrix} x & x^2 & 1/x^2 \\ 1 & 2x & -2/x^3 \\ 0 & 2 & 6/x^4 \end{vmatrix} = \frac{12}{x^2},$$

$$W_4 = \begin{vmatrix} x & x^2 & 1/x \\ 1 & 2x & -1/x^2 \\ 0 & 2 & 2/x^3 \end{vmatrix} = \frac{6}{x}.$$

Expanding W by cofactors of the last row yields

$$\begin{aligned} W &= -0W_1 + 0W_2 - \left(-\frac{6}{x^4}\right)W_3 + \left(-\frac{24}{x^5}\right)W_4 \\ &= \frac{6}{x^4} \frac{12}{x^2} - \frac{24}{x^5} \frac{6}{x} = -\frac{72}{x^6}. \end{aligned}$$

Since $F(x) = 12x^2$ and $P_0(x) = x^4$,

$$\frac{F}{P_0 W} = \frac{12x^2}{x^4} \left(-\frac{x^6}{72} \right) = -\frac{x^4}{6}.$$

Therefore, from (9.4.9),

$$u'_1 = -\left(-\frac{x^4}{6}\right) W_1 = \frac{x^4}{6} \left(-\frac{12}{x^4}\right) = -2,$$

$$u'_2 = -\frac{x^4}{6} W_2 = -\frac{x^4}{6} \left(-\frac{6}{x^5}\right) = \frac{1}{x},$$

$$u'_3 = -\left(-\frac{x^4}{6}\right) W_3 = \frac{x^4}{6} \frac{12}{x^2} = 2x^2,$$

$$u'_4 = -\frac{x^4}{6} W_4 = -\frac{x^4}{6} \frac{6}{x} = -x^3.$$

Integrating these and taking the constants of integration to be zero yields

$$u_1 = -2x, \quad u_2 = \ln|x|, \quad u_3 = \frac{2x^3}{3}, \quad u_4 = -\frac{x^4}{4}.$$

Hence,

$$\begin{aligned} y_p &= u_1 y_1 + u_2 y_2 + u_3 y_3 + u_4 y_4 \\ &= (-2x)x + (\ln|x|)x^2 + \frac{2x^3}{3} \frac{1}{x} + \left(-\frac{x^4}{4}\right) \frac{1}{x^2} \\ &= x^2 \ln|x| - \frac{19x^2}{12}. \end{aligned}$$

Since $-19x^2/12$ is a solution of the complementary equation, we redefine

$$y_p = x^2 \ln|x|.$$

Therefore

$$y = x^2 \ln|x| + c_1 x + c_2 x^2 + \frac{c_3}{x} + \frac{c_4}{x^2}$$

is the general solution of (9.4.10) on $(-\infty, 0)$ and $(0, \infty)$.

Section 10.1

Example 10.1.1 Tanks T_1 and T_2 contain 100 gallons and 300 gallons of salt solutions, respectively. Salt solutions are simultaneously added to both tanks from external sources, pumped from each tank to the other, and drained from both tanks (Figure 10.1.1). A solution with 1 pound of salt per gallon is pumped into T_1 from an external source at 5 gal/min, and a solution with 2 pounds of salt per gallon is pumped into T_2 from an external source at 4 gal/min. The solution from T_1 is pumped into T_2 at 2 gal/min, and the solution from T_2 is pumped into T_1 at 3 gal/min. T_1 is drained at 6 gal/min and T_2 is drained at 3 gal/min. Let $Q_1(t)$ and $Q_2(t)$ be the number of pounds of salt in T_1 and T_2 , respectively, at time $t > 0$. Derive a system of differential equations for Q_1 and Q_2 . Assume that both mixtures are well stirred.

Solution As in Section 4.2, let *rate in* and *rate out* denote the rates (lb/min) at which salt enters and leaves a tank; thus,

$$Q'_1 = (\text{rate in})_1 - (\text{rate out})_1,$$

$$Q'_2 = (\text{rate in})_2 - (\text{rate out})_2.$$

Note that the volumes of the solutions in T_1 and T_2 remain constant at 100 gallons and 300 gallons, respectively.

T_1 receives salt from the external source at the rate of

$$(1 \text{ lb/gal}) \times (5 \text{ gal/min}) = 5 \text{ lb/min},$$

and from T_2 at the rate of

$$(\text{lb/gal in } T_2) \times (3 \text{ gal/min}) = \frac{1}{300}Q_2 \times 3 = \frac{1}{100}Q_2 \text{ lb/min}.$$

Therefore

$$(\text{rate in})_1 = 5 + \frac{1}{100}Q_2. \quad (10.1.1)$$

Solution leaves T_1 at the rate of 8 gal/min, since 6 gal/min are drained and 2 gal/min are pumped to T_2 ; hence,

$$(\text{rate out})_1 = (\text{lb/gal in } T_1) \times (8 \text{ gal/min}) = \frac{1}{100}Q_1 \times 8 = \frac{2}{25}Q_1. \quad (10.1.2)$$

Eqns. (10.1.1) and (10.1.2) imply that

$$Q'_1 = 5 + \frac{1}{100}Q_2 - \frac{2}{25}Q_1. \quad (10.1.3)$$

T_2 receives salt from the external source at the rate of

$$(2 \text{ lb/gal}) \times (4 \text{ gal/min}) = 8 \text{ lb/min},$$

and from T_1 at the rate of

$$(\text{lb/gal in } T_1) \times (2 \text{ gal/min}) = \frac{1}{100}Q_1 \times 2 = \frac{1}{50}Q_1 \text{ lb/min}.$$

Therefore

$$(\text{rate in})_2 = 8 + \frac{1}{50}Q_1. \quad (10.1.4)$$

Solution leaves T_2 at the rate of 6 gal/min, since 3 gal/min are drained and 3 gal/min are pumped to T_1 ; hence,

$$(\text{rate out})_2 = (\text{lb/gal in } T_2) \times (6 \text{ gal/min}) = \frac{1}{300}Q_2 \times 6 = \frac{1}{50}Q_2. \quad (10.1.5)$$

Eqns. (10.1.4) and (10.1.5) imply that

$$Q'_2 = 8 + \frac{1}{50}Q_1 - \frac{1}{50}Q_2. \quad (10.1.6)$$

We say that (10.1.3) and (10.1.6) form a *system of two first order equations in two unknowns*, and write them together as

$$\begin{aligned} Q'_1 &= 5 - \frac{2}{25}Q_1 + \frac{1}{100}Q_2 \\ Q'_2 &= 8 + \frac{1}{50}Q_1 - \frac{1}{50}Q_2. \quad \blacksquare \end{aligned}$$

Example 10.1.2 A mass m_1 is suspended from a rigid support on a spring S_1 and a second mass m_2 is suspended from the first on a spring S_2 (Figure 10.1.2). The springs obey Hooke's law, with spring constants k_1 and k_2 . Internal friction causes the springs to exert damping forces proportional to the rates of change of their lengths, with damping constants c_1 and c_2 . Let $y_1 = y_1(t)$ and $y_2 = y_2(t)$ be the displacements of the two masses from their equilibrium positions at time t , measured positive upward. Derive a system of differential equations for y_1 and y_2 , assuming that the masses of the springs are negligible and that vertical external forces F_1 and F_2 also act on the objects.

Solution In equilibrium, S_1 supports both m_1 and m_2 and S_2 supports only m_2 . Therefore, if $\Delta\ell_1$ and $\Delta\ell_2$ are the elongations of the springs in equilibrium then

$$(m_1 + m_2)g = k_1\Delta\ell_1 \quad \text{and} \quad m_2g = k_2\Delta\ell_2. \quad (10.1.7)$$

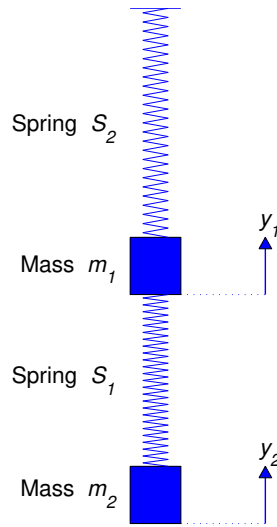


Figure 10.1.2

Let H_1 be the Hooke's law force acting on m_1 , and let D_1 be the damping force on m_1 . Similarly, let H_2 and D_2 be the Hooke's law and damping forces acting on m_2 . According to Newton's second law of motion,

$$\begin{aligned} m_1 y_1'' &= -m_1 g + H_1 + D_1 + F_1, \\ m_2 y_2'' &= -m_2 g + H_2 + D_2 + F_2. \end{aligned} \quad (10.1.8)$$

When the displacements are y_1 and y_2 , the change in length of S_1 is $-y_1 + \Delta\ell_1$ and the change in length of S_2 is $-y_2 + y_1 + \Delta\ell_2$. Both springs exert Hooke's law forces on m_1 , while only S_2 exerts a Hooke's law force on m_2 . These forces are in directions that tend to restore the springs to their natural lengths. Therefore

$$H_1 = k_1(-y_1 + \Delta\ell_1) - k_2(-y_2 + y_1 + \Delta\ell_2) \quad \text{and} \quad H_2 = k_2(-y_2 + y_1 + \Delta\ell_2). \quad (10.1.9)$$

When the velocities are y_1' and y_2' , S_1 and S_2 are changing length at the rates $-y_1'$ and $-y_2' + y_1'$, respectively. Both springs exert damping forces on m_1 , while only S_2 exerts a damping force on

m_2 . Since the force due to damping exerted by a spring is proportional to the rate of change of length of the spring and in a direction that opposes the change, it follows that

$$D_1 = -c_1 y_1' + c_2(y_2' - y_1') \quad \text{and} \quad D_2 = -c_2(y_2' - y_1'). \quad (10.1.10)$$

From (10.1.8), (10.1.9), and (10.1.10),

$$\begin{aligned} m_1 y_1'' &= -m_1 g + k_1(-y_1 + \Delta \ell_1) - k_2(-y_2 + y_1 + \Delta \ell_2) \\ &\quad -c_1 y_1' + c_2(y_2' - y_1') + F_1 \\ &= -(m_1 g - k_1 \Delta \ell_1 + k_2 \Delta \ell_2) - k_1 y_1 + k_2(y_2 - y_1) \\ &\quad -c_1 y_1' + c_2(y_2' - y_1') + F_1 \end{aligned} \quad (10.1.11)$$

and

$$\begin{aligned} m_2 y_2'' &= -m_2 g + k_2(-y_2 + y_1 + \Delta \ell_2) - c_2(y_2' - y_1') + F_2 \\ &= -(m_2 g - k_2 \Delta \ell_2) - k_2(y_2 - y_1) - c_2(y_2' - y_1') + F_2. \end{aligned} \quad (10.1.12)$$

From (10.1.7),

$$m_1 g - k_1 \Delta \ell_1 + k_2 \Delta \ell_2 = -m_2 g + k_2 \Delta \ell_2 = 0.$$

Therefore we can rewrite (10.1.11) and (10.1.12) as

$$\begin{aligned} m_1 y_1'' &= -(c_1 + c_2)y_1' + c_2 y_2' - (k_1 + k_2)y_1 + k_2 y_2 + F_1 \\ m_2 y_2'' &= c_2 y_1' - c_2 y_2' + k_2 y_1 - k_2 y_2 + F_2. \quad \blacksquare \end{aligned}$$

Example 10.1.3 Let $\mathbf{X} = \mathbf{X}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ be the position vector at time t of an object with mass m , relative to a rectangular coordinate system with origin at Earth's center (Figure 10.1.3). According to Newton's law of gravitation, Earth's gravitational force $\mathbf{F} = \mathbf{F}(x, y, z)$ on the object is inversely proportional to the square of the distance of the object from Earth's center, and directed toward the center; thus,

$$\mathbf{F} = \frac{K}{\|\mathbf{X}\|^2} \left(-\frac{\mathbf{X}}{\|\mathbf{X}\|} \right) = -K \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}, \quad (10.1.13)$$

where K is a constant. To determine K , we observe that the magnitude of $\mathbf{F}$ is

$$\|\mathbf{F}\| = K \frac{\|\mathbf{X}\|}{\|\mathbf{X}\|^3} = \frac{K}{\|\mathbf{X}\|^2} = \frac{K}{(x^2 + y^2 + z^2)}.$$

Let R be Earth's radius. Since $\|\mathbf{F}\| = mg$ when the object is at Earth's surface,

$$mg = \frac{K}{R^2}, \quad \text{so} \quad K = mgR^2.$$

Therefore we can rewrite (10.1.13) as

$$\mathbf{F} = -mgR^2 \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}.$$

Now suppose $\mathbf{F}$ is the only force acting on the object. According to Newton's second law of motion, $\mathbf{F} = m\mathbf{X}''$; that is,

$$m(x''\mathbf{i} + y''\mathbf{j} + z''\mathbf{k}) = -mgR^2 \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}.$$

Cancelling the common factor m and equating components on the two sides of this equation yields the system

$$\begin{aligned}x'' &= -\frac{gR^2x}{(x^2 + y^2 + z^2)^{3/2}} \\y'' &= -\frac{gR^2y}{(x^2 + y^2 + z^2)^{3/2}} \\z'' &= -\frac{gR^2z}{(x^2 + y^2 + z^2)^{3/2}}.\end{aligned}\tag{10.1.14}$$

Example 10.1.4 Rewrite the system

$$\begin{aligned}m_1y_1'' &= -(c_1 + c_2)y_1' + c_2y_2' - (k_1 + k_2)y_1 + k_2y_2 + F_1 \\m_2y_2'' &= c_2y_1' - c_2y_2' + k_2y_1 - k_2y_2 + F_2.\end{aligned}\tag{10.1.16}$$

derived in Example 10.1.2 as a system of first order equations.

Solution If we define $v_1 = y_1'$ and $v_2 = y_2'$, then $v_1' = y_1''$ and $v_2' = y_2''$, so (10.1.16) becomes

$$\begin{aligned}m_1v_1' &= -(c_1 + c_2)v_1 + c_2v_2 - (k_1 + k_2)y_1 + k_2y_2 + F_1 \\m_2v_2' &= c_2v_1 - c_2v_2 + k_2y_1 - k_2y_2 + F_2.\end{aligned}$$

Therefore $\{y_1, y_2, v_1, v_2\}$ satisfies the 4×4 first order system

$$\begin{aligned}y_1' &= v_1 \\y_2' &= v_2 \\v_1' &= \frac{1}{m_1} [-(c_1 + c_2)v_1 + c_2v_2 - (k_1 + k_2)y_1 + k_2y_2 + F_1] \\v_2' &= \frac{1}{m_2} [c_2v_1 - c_2v_2 + k_2y_1 - k_2y_2 + F_2].\end{aligned}\tag{10.1.17}$$

Remark: The difference in form between (10.1.15) and (10.1.17), due to the way in which the unknowns are *denoted* in the two systems, isn't important; (10.1.17) is a first order system, in that each equation in (10.1.17) expresses the first derivative of one of the unknown functions in a way that does not involve derivatives of any of the other unknowns.

Example 10.1.5 Rewrite the system

$$\begin{aligned}x'' &= f(t, x, x', y, y', y'') \\y''' &= g(t, x, x', y, y', y'')\end{aligned}$$

as a first order system.

Solution We regard x, x', y, y' , and y'' as unknown functions, and rename them

$$x = x_1, \quad x' = x_2, \quad y = y_1, \quad y' = y_2, \quad y'' = y_3.$$

These unknowns satisfy the system

$$\begin{aligned}x_1' &= x_2 \\x_2' &= f(t, x_1, x_2, y_1, y_2, y_3) \\y_1' &= y_2 \\y_2' &= y_3 \\y_3' &= g(t, x_1, x_2, y_1, y_2, y_3).\end{aligned}$$

Example 10.1.6 Rewrite the equation

$$y^{(4)} + 4y''' + 6y'' + 4y' + y = 0 \quad (10.1.18)$$

as a 4×4 first order system.

Solution We regard y , y' , y'' , and y''' as unknowns and rename them

$$y = y_1, \quad y' = y_2, \quad y'' = y_3, \quad \text{and} \quad y''' = y_4.$$

Then $y^{(4)} = y_4'$, so (10.1.18) can be written as

$$y_4' + 4y_4 + 6y_3 + 4y_2 + y_1 = 0.$$

Therefore $\{y_1, y_2, y_3, y_4\}$ satisfies the system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= y_4 \\ y_4' &= -4y_4 - 6y_3 - 4y_2 - y_1. \quad \blacksquare \end{aligned}$$

Example 10.1.7 Rewrite

$$x''' = f(t, x, x', x'')$$

as a system of first order equations.

Solution We regard x , x' , and x'' as unknowns and rename them

$$x = y_1, \quad x' = y_2, \quad \text{and} \quad x'' = y_3.$$

Then

$$y_1' = x' = y_2, \quad y_2' = x'' = y_3, \quad \text{and} \quad y_3' = x'''.$$

Therefore $\{y_1, y_2, y_3\}$ satisfies the first order system

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= y_3 \\ y_3' &= f(t, y_1, y_2, y_3). \end{aligned}$$

Section 10.2

Example 10.2.1

(a) Write the system

$$\begin{aligned} y_1' &= y_1 + 2y_2 + 2e^{4t} \\ y_2' &= 2y_1 + y_2 + e^{4t} \end{aligned} \quad (10.2.3)$$

in matrix form and conclude from Theorem 10.2.1 that every initial value problem for (10.2.3) has a unique solution on $(-\infty, \infty)$.

(b) Verify that

$$\mathbf{y} = \frac{1}{5} \begin{bmatrix} 8 \\ 7 \end{bmatrix} e^{4t} + c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} \quad (10.2.4)$$

is a solution of (10.2.3) for all values of the constants c_1 and c_2 .

(c) Find the solution of the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t}, \quad \mathbf{y}(0) = \frac{1}{5} \begin{bmatrix} 3 \\ 22 \end{bmatrix}. \quad (10.2.5)$$

Solution (a) The system (10.2.3) can be written in matrix form as

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t}.$$

An initial value problem for (10.2.3) can be written as

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t}, \quad \mathbf{y}(t_0) = \begin{bmatrix} k_1 \\ k_2 \end{bmatrix}.$$

Since the coefficient matrix and the forcing function are both continuous on $(-\infty, \infty)$, Theorem 10.2.1 implies that this problem has a unique solution on $(-\infty, \infty)$.

Solution (b) If $\mathbf{y}$ is given by (10.2.4), then

$$\begin{aligned} A\mathbf{y} + \mathbf{f} &= \frac{1}{5} \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 8 \\ 7 \end{bmatrix} e^{4t} + c_1 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} \\ &\quad + c_2 \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t} \\ &= \frac{1}{5} \begin{bmatrix} 22 \\ 23 \end{bmatrix} e^{4t} + c_1 \begin{bmatrix} 3 \\ 3 \end{bmatrix} e^{3t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-t} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{4t} \\ &= \frac{1}{5} \begin{bmatrix} 32 \\ 28 \end{bmatrix} e^{4t} + 3c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} - c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t} = \mathbf{y}'. \end{aligned}$$

Solution (c) We must choose c_1 and c_2 in (10.2.4) so that

$$\frac{1}{5} \begin{bmatrix} 8 \\ 7 \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 3 \\ 22 \end{bmatrix},$$

which is equivalent to

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}.$$

Solving this system yields $c_1 = 1$, $c_2 = -2$, so

$$\mathbf{y} = \frac{1}{5} \begin{bmatrix} 8 \\ 7 \end{bmatrix} e^{4t} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t} - 2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$$

is the solution of (10.2.5).

Section 10.3

Example 10.3.1 Show that the vector functions

$$\mathbf{y}_1 = \begin{bmatrix} e^t \\ 0 \\ e^{-t} \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} 0 \\ e^{3t} \\ 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{y}_3 = \begin{bmatrix} e^{2t} \\ e^{3t} \\ 0 \end{bmatrix}$$

are linearly independent on every interval (a, b) .

Solution Suppose

$$c_1 \begin{bmatrix} e^t \\ 0 \\ e^{-t} \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ e^{3t} \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} e^{2t} \\ e^{3t} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad a < t < b.$$

We must show that $c_1 = c_2 = c_3 = 0$. Rewriting this equation in matrix form yields

$$\begin{bmatrix} e^t & 0 & e^{2t} \\ 0 & e^{3t} & e^{3t} \\ e^{-t} & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad a < t < b.$$

Expanding the determinant of this system in cofactors of the entries of the first row yields

$$\begin{aligned} \begin{vmatrix} e^t & 0 & e^{2t} \\ 0 & e^{3t} & e^{3t} \\ e^{-t} & 1 & 0 \end{vmatrix} &= e^t \begin{vmatrix} e^{3t} & e^{3t} \\ e^{-t} & 0 \end{vmatrix} - 0 \begin{vmatrix} 0 & e^{3t} \\ e^{-t} & 1 \end{vmatrix} + e^{2t} \begin{vmatrix} 0 & e^{3t} \\ e^{-t} & 1 \end{vmatrix} \\ &= e^t(-e^{3t}) + e^{2t}(-e^{2t}) = -2e^{4t}. \end{aligned}$$

Since this determinant is never zero, $c_1 = c_2 = c_3 = 0$. ■

Example 10.3.2 The vector functions

$$\mathbf{y}_1 = \begin{bmatrix} -e^{2t} \\ 2e^{2t} \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix}$$

are solutions of the constant coefficient system

$$\mathbf{y}' = \begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix} \mathbf{y} \tag{10.3.7}$$

on $(-\infty, \infty)$. (Verify.)

- (a) Compute the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2\}$ directly from the definition (10.3.5)
- (b) Verify Abel's formula (10.3.6) for the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2\}$.
- (c) Find the general solution of (10.3.7).
- (d) Solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 4 \\ -5 \end{bmatrix}. \tag{10.3.8}$$

Solution (a) From (10.3.5)

$$W(t) = \begin{vmatrix} -e^{2t} & -e^{-t} \\ 2e^{2t} & e^{-t} \end{vmatrix} = e^{2t}e^{-t}((-1)(1) - (-1)(2)) = e^t. \quad (10.3.9)$$

Solution (b) Here

$$A = \begin{bmatrix} -4 & -3 \\ 6 & 5 \end{bmatrix},$$

so $\text{tr}(A) = -4 + 5 = 1$. If t_0 is an arbitrary real number then (10.3.6) implies that

$$W(t) = W(t_0) \exp\left(\int_{t_0}^t 1 \, ds\right) = \begin{vmatrix} -e^{2t_0} & -e^{-t_0} \\ 2e^{2t_0} & e^{-t_0} \end{vmatrix} e^{(t-t_0)} = e^{t_0} e^{t-t_0} = e^t,$$

which is consistent with (10.3.9).

Solution (c) Since $W(t) \neq 0$, Theorem 10.3.3 implies that $\{\mathbf{y}_1, \mathbf{y}_2\}$ is a fundamental set of solutions of (10.3.7) and

$$Y = \begin{bmatrix} -e^{2t} & -e^{-t} \\ 2e^{2t} & e^{-t} \end{bmatrix}$$

is a fundamental matrix for (10.3.7). Therefore the general solution of (10.3.7) is

$$\mathbf{y} = c_1 \mathbf{y}_1 + c_2 \mathbf{y}_2 = c_1 \begin{bmatrix} -e^{2t} \\ 2e^{2t} \end{bmatrix} + c_2 \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix} = \begin{bmatrix} -e^{2t} & -e^{-t} \\ 2e^{2t} & e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \quad (10.3.10)$$

Solution (d) Setting $t = 0$ in (10.3.10) and imposing the initial condition in (10.3.8) yields

$$c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}.$$

Thus,

$$\begin{aligned} -c_1 - c_2 &= 4 \\ 2c_1 + c_2 &= -5. \end{aligned}$$

The solution of this system is $c_1 = -1$, $c_2 = -3$. Substituting these values into (10.3.10) yields

$$\mathbf{y} = - \begin{bmatrix} -e^{2t} \\ 2e^{2t} \end{bmatrix} - 3 \begin{bmatrix} -e^{-t} \\ e^{-t} \end{bmatrix} = \begin{bmatrix} e^{2t} + 3e^{-t} \\ -2e^{2t} - 3e^{-t} \end{bmatrix}$$

as the solution of (10.3.8).

Section 10.4

Example 10.4.1

(a) Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} \mathbf{y}. \quad (10.4.8)$$

(b) Solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 2 & 4 \\ 4 & 2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 5 \\ -1 \end{bmatrix}. \quad (10.4.9)$$

Solution (a) The characteristic polynomial of the coefficient matrix A in (10.4.8) is

$$\begin{aligned} \begin{vmatrix} 2-\lambda & 4 \\ 4 & 2-\lambda \end{vmatrix} &= (\lambda-2)^2 - 16 \\ &= (\lambda-2-4)(\lambda-2+4) \\ &= (\lambda-6)(\lambda+2). \end{aligned}$$

Hence, $\lambda_1 = 6$ and $\lambda_2 = -2$ are eigenvalues of A . To obtain the eigenvectors, we must solve the system

$$\begin{bmatrix} 2-\lambda & 4 \\ 4 & 2-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad (10.4.10)$$

with $\lambda = 6$ and $\lambda = -2$. Setting $\lambda = 6$ in (10.4.10) yields

$$\begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which implies that $x_1 = x_2$. Taking $x_2 = 1$ yields the eigenvector

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

so

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t}$$

is a solution of (10.4.8). Setting $\lambda = -2$ in (10.4.10) yields

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

which implies that $x_1 = -x_2$. Taking $x_2 = 1$ yields the eigenvector

$$\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix},$$

so

$$\mathbf{y}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}$$

is a solution of (10.4.8). From Theorem 10.4.1, the general solution of (10.4.8) is

$$\mathbf{y} = c_1 \mathbf{y}_1 + c_2 \mathbf{y}_2 = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t}. \quad (10.4.11)$$

Solution (b) To satisfy the initial condition in (10.4.9), we must choose c_1 and c_2 in (10.4.11) so that

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix}.$$

This is equivalent to the system

$$\begin{aligned} c_1 - c_2 &= 5 \\ c_1 + c_2 &= -1, \end{aligned}$$

so $c_1 = 2$, $c_2 = -3$. Therefore the solution of (10.4.9) is

$$\mathbf{y} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{6t} - 3 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{-2t},$$

or, in terms of components,

$$y_1 = 2e^{6t} + 3e^{-2t}, \quad y_2 = 2e^{6t} - 3e^{-2t}.$$

Example 10.4.2

(a) Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{bmatrix} \mathbf{y}. \quad (10.4.12)$$

(b) Solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 3 & -1 & -1 \\ -2 & 3 & 2 \\ 4 & -1 & -2 \end{bmatrix} \mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 2 \\ -1 \\ 8 \end{bmatrix}. \quad (10.4.13)$$

Solution (a) The characteristic polynomial of the coefficient matrix A in (10.4.12) is

$$\begin{vmatrix} 3 - \lambda & -1 & -1 \\ -2 & 3 - \lambda & 2 \\ 4 & -1 & -2 - \lambda \end{vmatrix} = -(\lambda - 2)(\lambda - 3)(\lambda + 1).$$

Hence, the eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = 3$, and $\lambda_3 = -1$. To find the eigenvectors, we must solve the system

$$\begin{bmatrix} 3 - \lambda & -1 & -1 \\ -2 & 3 - \lambda & 2 \\ 4 & -1 & -2 - \lambda \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad (10.4.14)$$

with $\lambda = 2, 3, -1$. With $\lambda = 2$, the augmented matrix of (10.4.14) is

$$\left[\begin{array}{cccc|c} 1 & -1 & -1 & \vdots & 0 \\ -2 & 1 & 2 & \vdots & 0 \\ 4 & -1 & -4 & \vdots & 0 \end{array} \right],$$

which is row equivalent to

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & \vdots & 0 \\ 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array} \right].$$

Hence, $x_1 = x_3$ and $x_2 = 0$. Taking $x_3 = 1$ yields

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t}$$

as a solution of (10.4.12). With $\lambda = 3$, the augmented matrix of (10.4.14) is

$$\begin{bmatrix} 0 & -1 & -1 & \vdots & 0 \\ -2 & 0 & 2 & \vdots & 0 \\ 4 & -1 & -5 & \vdots & 0 \end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix} 1 & 0 & -1 & \vdots & 0 \\ 0 & 1 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Hence, $x_1 = x_3$ and $x_2 = -x_3$. Taking $x_3 = 1$ yields

$$\mathbf{y}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{3t}$$

as a solution of (10.4.12). With $\lambda = -1$, the augmented matrix of (10.4.14) is

$$\begin{bmatrix} 4 & -1 & -1 & \vdots & 0 \\ -2 & 4 & 2 & \vdots & 0 \\ 4 & -1 & -1 & \vdots & 0 \end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix} 1 & 0 & -\frac{1}{7} & \vdots & 0 \\ 0 & 1 & \frac{3}{7} & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Hence, $x_1 = x_3/7$ and $x_2 = -3x_3/7$. Taking $x_3 = 7$ yields

$$\mathbf{y}_3 = \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} e^{-t}$$

as a solution of (10.4.12). By Theorem 10.4.1, the general solution of (10.4.12) is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{3t} + c_3 \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} e^{-t},$$

which can also be written as

$$\mathbf{y} = \begin{bmatrix} e^{2t} & e^{3t} & e^{-t} \\ 0 & -e^{3t} & -3e^{-t} \\ e^{2t} & e^{3t} & 7e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}. \quad (10.4.15)$$

Solution (b) To satisfy the initial condition in (10.4.13) we must choose c_1, c_2, c_3 in (10.4.15) so that

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -3 \\ 1 & 1 & 7 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 8 \end{bmatrix}.$$

Solving this system yields $c_1 = 3, c_2 = -2, c_3 = 1$. Hence, the solution of (10.4.13) is

$$\begin{aligned} \mathbf{y} &= \begin{bmatrix} e^{2t} & e^{3t} & e^{-t} \\ 0 & -e^{3t} & -3e^{-t} \\ e^{2t} & e^{3t} & 7e^{-t} \end{bmatrix} \begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \\ &= 3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} - 2 \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{3t} + \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} e^{-t}. \end{aligned}$$

Example 10.4.3 Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix} \mathbf{y}. \quad (10.4.16)$$

Solution The characteristic polynomial of the coefficient matrix A in (10.4.16) is

$$\begin{vmatrix} -3 - \lambda & 2 & 2 \\ 2 & -3 - \lambda & 2 \\ 2 & 2 & -3 - \lambda \end{vmatrix} = -(\lambda - 1)(\lambda + 5)^2.$$

Hence, $\lambda_1 = 1$ is an eigenvalue of multiplicity 1, while $\lambda_2 = -5$ is an eigenvalue of multiplicity 2. Eigenvectors associated with $\lambda_1 = 1$ are solutions of the system with augmented matrix

$$\begin{bmatrix} -4 & 2 & 2 & \vdots & 0 \\ 2 & -4 & 2 & \vdots & 0 \\ 2 & 2 & -4 & \vdots & 0 \end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix} 1 & 0 & -1 & \vdots & 0 \\ 0 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Hence, $x_1 = x_2 = x_3$, and we choose $x_3 = 1$ to obtain the solution

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^t \quad (10.4.17)$$

of (10.4.16). Eigenvectors associated with $\lambda_2 = -5$ are solutions of the system with augmented matrix

$$\begin{bmatrix} 2 & 2 & 2 & \vdots & 0 \\ 2 & 2 & 2 & \vdots & 0 \\ 2 & 2 & 2 & \vdots & 0 \end{bmatrix}.$$

Hence, the components of these eigenvectors need only satisfy the single condition

$$x_1 + x_2 + x_3 = 0.$$

Since there's only one equation here, we can choose x_2 and x_3 arbitrarily. We obtain one eigenvector by choosing $x_2 = 0$ and $x_3 = 1$, and another by choosing $x_2 = 1$ and $x_3 = 0$. In both cases $x_1 = -1$. Therefore

$$\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

are linearly independent eigenvectors associated with $\lambda_2 = -5$, and the corresponding solutions of (10.4.16) are

$$\mathbf{y}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-5t} \quad \text{and} \quad \mathbf{y}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-5t}.$$

Because of this and (10.4.17), Theorem 10.4.1 implies that the general solution of (10.4.16) is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} e^{-5t} + c_3 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-5t}.$$

Section 10.5

Example 10.5.1 Show that the system

$$\mathbf{y}' = \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix} \mathbf{y} \tag{10.5.1}$$

does not have a fundamental set of solutions of the form $\{\mathbf{x}_1 e^{\lambda_1 t}, \mathbf{x}_2 e^{\lambda_2 t}\}$, where λ_1 and λ_2 are eigenvalues of the coefficient matrix A of (10.5.1) and $\mathbf{x}_1$ and $\mathbf{x}_2$ are associated linearly independent eigenvectors.

Solution The characteristic polynomial of A is

$$\begin{aligned} \begin{vmatrix} 11 - \lambda & -25 \\ 4 & -9 - \lambda \end{vmatrix} &= (\lambda - 11)(\lambda + 9) + 100 \\ &= \lambda^2 - 2\lambda + 1 = (\lambda - 1)^2. \end{aligned}$$

Hence, $\lambda = 1$ is the only eigenvalue of A . The augmented matrix of the system $(A - I)\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} 10 & -25 & \vdots & 0 \\ 4 & -10 & \vdots & 0 \end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix} 1 & -\frac{5}{2} & \vdots & 0 \\ 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Hence, $x_1 = 5x_2/2$ where x_2 is arbitrary. Therefore all eigenvectors of A are scalar multiples of $\mathbf{x}_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$, so A does not have a set of two linearly independent eigenvectors. ■

Example 10.5.2 Use Theorem 10.5.1 to find the general solution of the system

$$\mathbf{y}' = \begin{bmatrix} 11 & -25 \\ 4 & -9 \end{bmatrix} \mathbf{y} \quad (10.5.6)$$

considered in Example 10.5.1.

Solution In Example 10.5.1 we saw that $\lambda_1 = 1$ is an eigenvalue of multiplicity 2 of the coefficient matrix A in (10.5.6), and that all of the eigenvectors of A are multiples of

$$\mathbf{x} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}.$$

Therefore

$$\mathbf{y}_1 = \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^t$$

is a solution of (10.5.6). From Theorem 10.5.1, a second solution is given by $\mathbf{y}_2 = \mathbf{u}e^t + \mathbf{x}te^t$, where $(A - I)\mathbf{u} = \mathbf{x}$. The augmented matrix of this system is

$$\left[\begin{array}{cc|c} 10 & -25 & 5 \\ 4 & -10 & 2 \end{array} \right],$$

which is row equivalent to

$$\left[\begin{array}{cc|c} 1 & -\frac{5}{2} & \frac{1}{2} \\ 0 & 0 & 0 \end{array} \right].$$

Therefore the components of $\mathbf{u}$ must satisfy

$$u_1 - \frac{5}{2}u_2 = \frac{1}{2},$$

where u_2 is arbitrary. We choose $u_2 = 0$, so that $u_1 = 1/2$ and

$$\mathbf{u} = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}.$$

Thus,

$$\mathbf{y}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{e^t}{2} + \begin{bmatrix} 5 \\ 2 \end{bmatrix} te^t.$$

Since $\mathbf{y}_1$ and $\mathbf{y}_2$ are linearly independent by Theorem 10.5.1, they form a fundamental set of solutions of (10.5.6). Therefore the general solution of (10.5.6) is

$$\mathbf{y} = c_1 \begin{bmatrix} 5 \\ 2 \end{bmatrix} e^t + c_2 \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \frac{e^t}{2} + \begin{bmatrix} 5 \\ 2 \end{bmatrix} te^t \right). \blacksquare$$

Note that choosing the arbitrary constant u_2 to be nonzero is equivalent to adding a scalar multiple of $\mathbf{y}_1$ to the second solution $\mathbf{y}_2$ (Exercise 10.5.33).

Example 10.5.3 Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 3 & 4 & -10 \\ 2 & 1 & -2 \\ 2 & 2 & -5 \end{bmatrix} \mathbf{y}. \quad (10.5.7)$$

Solution The characteristic polynomial of the coefficient matrix A in (10.5.7) is

$$\begin{vmatrix} 3-\lambda & 4 & -10 \\ 2 & 1-\lambda & -2 \\ 2 & 2 & -5-\lambda \end{vmatrix} = -(\lambda-1)(\lambda+1)^2.$$

Hence, the eigenvalues are $\lambda_1 = 1$ with multiplicity 1 and $\lambda_2 = -1$ with multiplicity 2.

Eigenvectors associated with $\lambda_1 = 1$ must satisfy $(A - I)\mathbf{x} = \mathbf{0}$. The augmented matrix of this system is

$$\left[\begin{array}{ccc|c} 2 & 4 & -10 & 0 \\ 2 & 0 & -2 & 0 \\ 2 & 2 & -6 & 0 \end{array} \right],$$

which is row equivalent to

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

Hence, $x_1 = x_3$ and $x_2 = 2x_3$, where x_3 is arbitrary. Choosing $x_3 = 1$ yields the eigenvector

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Therefore

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^t$$

is a solution of (10.5.7).

Eigenvectors associated with $\lambda_2 = -1$ satisfy $(A + I)\mathbf{x} = \mathbf{0}$. The augmented matrix of this system is

$$\left[\begin{array}{ccc|c} 4 & 4 & -10 & 0 \\ 2 & 2 & -2 & 0 \\ 2 & 2 & -4 & 0 \end{array} \right],$$

which is row equivalent to

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

Hence, $x_3 = 0$ and $x_1 = -x_2$, where x_2 is arbitrary. Choosing $x_2 = 1$ yields the eigenvector

$$\mathbf{x}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix},$$

so

$$\mathbf{y}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-t}$$

is a solution of (10.5.7).

Since all the eigenvectors of A associated with $\lambda_2 = -1$ are multiples of $\mathbf{x}_2$, we must now use Theorem 10.5.1 to find a third solution of (10.5.7) in the form

$$\mathbf{y}_3 = \mathbf{u}e^{-t} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} te^{-t}, \quad (10.5.8)$$

where $\mathbf{u}$ is a solution of $(A + I)\mathbf{u} = \mathbf{x}_2$. The augmented matrix of this system is

$$\begin{bmatrix} 4 & 4 & -10 & \vdots & -1 \\ 2 & 2 & -2 & \vdots & 1 \\ 2 & 2 & -4 & \vdots & 0 \end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix} 1 & 1 & 0 & \vdots & 1 \\ 0 & 0 & 1 & \vdots & \frac{1}{2} \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Hence, $u_3 = 1/2$ and $u_1 = 1 - u_2$, where u_2 is arbitrary. Choosing $u_2 = 0$ yields

$$\mathbf{u} = \begin{bmatrix} 1 \\ 0 \\ \frac{1}{2} \end{bmatrix},$$

and substituting this into (10.5.8) yields the solution

$$\mathbf{y}_3 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \frac{e^{-t}}{2} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} te^{-t}$$

of (10.5.7).

Since the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ at $t = 0$ is

$$\begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & 0 \\ 1 & 0 & \frac{1}{2} \end{vmatrix} = \frac{1}{2},$$

$\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is a fundamental set of solutions of (10.5.7). Therefore the general solution of (10.5.7) is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} e^{-t} + c_3 \left(\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \frac{e^{-t}}{2} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} te^{-t} \right).$$

Example 10.5.4 Use Theorem 10.5.2 to find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 3 & -1 \\ 0 & 2 & 2 \end{bmatrix} \mathbf{y}. \quad (10.5.11)$$

Solution The characteristic polynomial of the coefficient matrix A in (10.5.11) is

$$\begin{vmatrix} 1-\lambda & 1 & 1 \\ 1 & 3-\lambda & -1 \\ 0 & 2 & 2-\lambda \end{vmatrix} = -(\lambda-2)^3.$$

Hence, $\lambda_1 = 2$ is an eigenvalue of multiplicity 3. The associated eigenvectors satisfy $(A - 2I)\mathbf{x} = \mathbf{0}$. The augmented matrix of this system is

$$\left[\begin{array}{cccc|c} -1 & 1 & 1 & \vdots & 0 \\ 1 & 1 & -1 & \vdots & 0 \\ 0 & 2 & 0 & \vdots & 0 \end{array} \right],$$

which is row equivalent to

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & \vdots & 0 \\ 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array} \right].$$

Hence, $x_1 = x_3$ and $x_2 = 0$, so the eigenvectors are all scalar multiples of

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Therefore

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t}$$

is a solution of (10.5.11).

We now find a second solution of (10.5.11) in the form

$$\mathbf{y}_2 = \mathbf{u}e^{2t} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} te^{2t},$$

where $\mathbf{u}$ satisfies $(A - 2I)\mathbf{u} = \mathbf{x}_1$. The augmented matrix of this system is

$$\left[\begin{array}{cccc|c} -1 & 1 & 1 & \vdots & 1 \\ 1 & 1 & -1 & \vdots & 0 \\ 0 & 2 & 0 & \vdots & 1 \end{array} \right],$$

which is row equivalent to

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & \vdots & -\frac{1}{2} \\ 0 & 1 & 0 & \vdots & \frac{1}{2} \\ 0 & 0 & 0 & \vdots & 0 \end{array} \right].$$

Letting $u_3 = 0$ yields $u_1 = -1/2$ and $u_2 = 1/2$; hence,

$$\mathbf{u} = \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

and

$$\mathbf{y}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \frac{e^{2t}}{2} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} t e^{2t}$$

is a solution of (10.5.11).

We now find a third solution of (10.5.11) in the form

$$\mathbf{y}_3 = \mathbf{v} e^{2t} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \frac{t e^{2t}}{2} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \frac{t^2 e^{2t}}{2}$$

where $\mathbf{v}$ satisfies $(A - 2I)\mathbf{v} = \mathbf{u}$. The augmented matrix of this system is

$$\left[\begin{array}{cccc|c} -1 & 1 & 1 & \vdots & -\frac{1}{2} \\ 1 & 1 & -1 & \vdots & \frac{1}{2} \\ 0 & 2 & 0 & \vdots & 0 \end{array} \right],$$

which is row equivalent to

$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & \vdots & \frac{1}{2} \\ 0 & 1 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{array} \right].$$

Letting $v_3 = 0$ yields $v_1 = 1/2$ and $v_2 = 0$; hence,

$$\mathbf{v} = \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore

$$\mathbf{y}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{e^{2t}}{2} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \frac{t e^{2t}}{2} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \frac{t^2 e^{2t}}{2}$$

is a solution of (10.5.11). Since $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ are linearly independent by Theorem 10.5.2, they form a fundamental set of solutions of (10.5.11). Therefore the general solution of (10.5.11) is

$$\begin{aligned} \mathbf{y} = & c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^{2t} + c_2 \left(\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \frac{e^{2t}}{2} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} t e^{2t} \right) \\ & + c_3 \left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \frac{e^{2t}}{2} + \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \frac{t e^{2t}}{2} + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \frac{t^2 e^{2t}}{2} \right). \end{aligned}$$

Example 10.5.5 Use Theorem 10.5.3 to find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 0 & 0 & 1 \\ -1 & 1 & 1 \\ -1 & 0 & 2 \end{bmatrix} \mathbf{y}. \quad (10.5.15)$$

Solution The characteristic polynomial of the coefficient matrix A in (10.5.15) is

$$\begin{vmatrix} -\lambda & 0 & 1 \\ -1 & 1-\lambda & 1 \\ -1 & 0 & 2-\lambda \end{vmatrix} = -(\lambda-1)^3.$$

Hence, $\lambda_1 = 1$ is an eigenvalue of multiplicity 3. The associated eigenvectors satisfy $(A-I)\mathbf{x} = \mathbf{0}$. The augmented matrix of this system is

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 \end{array} \right],$$

which is row equivalent to

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

Hence, $x_1 = x_3$ and x_2 is arbitrary, so the eigenvectors are of the form

$$\mathbf{x} = \begin{bmatrix} x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

Therefore the vectors

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad \text{and} \quad \mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{10.5.16}$$

form a basis for the eigenspace, and

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^t \quad \text{and} \quad \mathbf{y}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^t$$

are linearly independent solutions of (10.5.15).

To find a third linearly independent solution of (10.5.15), we must find constants α and β (not both zero) such that the system

$$(A-I)\mathbf{u} = \alpha\mathbf{x}_1 + \beta\mathbf{x}_2 \tag{10.5.17}$$

has a solution $\mathbf{u}$. The augmented matrix of this system is

$$\left[\begin{array}{ccc|c} -1 & 0 & 1 & \alpha \\ -1 & 0 & 1 & \beta \\ -1 & 0 & 1 & \alpha \end{array} \right],$$

which is row equivalent to

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -\alpha \\ 0 & 0 & 0 & \beta - \alpha \\ 0 & 0 & 0 & 0 \end{array} \right]. \tag{10.5.18}$$

Therefore (10.5.17) has a solution if and only if $\beta = \alpha$, where α is arbitrary. If $\alpha = \beta = 1$ then (10.5.12) and (10.5.16) yield

$$\mathbf{x}_3 = \mathbf{x}_1 + \mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix},$$

and the augmented matrix (10.5.18) becomes

$$\begin{bmatrix} 1 & 0 & -1 & \vdots & -1 \\ 0 & 0 & 0 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

This implies that $u_1 = -1 + u_3$, while u_2 and u_3 are arbitrary. Choosing $u_2 = u_3 = 0$ yields

$$\mathbf{u} = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore (10.5.14) implies that

$$\mathbf{y}_3 = \mathbf{u}e^t + \mathbf{x}_3te^t = \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}e^t + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}te^t$$

is a solution of (10.5.15). Since $\mathbf{y}_1$, $\mathbf{y}_2$, and $\mathbf{y}_3$ are linearly independent by Theorem 10.5.3, they form a fundamental set of solutions for (10.5.15). Therefore the general solution of (10.5.15) is

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} e^t + c_3 \left(\begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} e^t + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} te^t \right). \quad \blacksquare$$

Section 10.6

Example 10.6.1 Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 4 & -5 \\ 5 & -2 \end{bmatrix} \mathbf{y}. \quad (10.6.6)$$

Solution The characteristic polynomial of the coefficient matrix A in (10.6.6) is

$$\begin{vmatrix} 4 - \lambda & -5 \\ 5 & -2 - \lambda \end{vmatrix} = (\lambda - 1)^2 + 16.$$

Hence, $\lambda = 1 + 4i$ is an eigenvalue of A . The associated eigenvectors satisfy $(A - (1 + 4i)I)\mathbf{x} = \mathbf{0}$. The augmented matrix of this system is

$$\begin{bmatrix} 3 - 4i & -5 & \vdots & 0 \\ 5 & -3 - 4i & \vdots & 0 \end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix} 1 & -\frac{3+4i}{5} & \vdots & 0 \\ 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Therefore $x_1 = (3 + 4i)x_2/5$. Taking $x_2 = 5$ yields $x_1 = 3 + 4i$, so

$$\mathbf{x} = \begin{bmatrix} 3 + 4i \\ 5 \end{bmatrix}$$

is an eigenvector. The real and imaginary parts of

$$e^t(\cos 4t + i \sin 4t) \begin{bmatrix} 3 + 4i \\ 5 \end{bmatrix}$$

are

$$\mathbf{y}_1 = e^t \begin{bmatrix} 3 \cos 4t - 4 \sin 4t \\ 5 \cos 4t \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = e^t \begin{bmatrix} 3 \sin 4t + 4 \cos 4t \\ 5 \sin 4t \end{bmatrix},$$

which are linearly independent solutions of (10.6.6). The general solution of (10.6.6) is

$$\mathbf{y} = c_1 e^t \begin{bmatrix} 3 \cos 4t - 4 \sin 4t \\ 5 \cos 4t \end{bmatrix} + c_2 e^t \begin{bmatrix} 3 \sin 4t + 4 \cos 4t \\ 5 \sin 4t \end{bmatrix}.$$

Example 10.6.2 Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} -14 & 39 \\ -6 & 16 \end{bmatrix} \mathbf{y}. \quad (10.6.7)$$

Solution The characteristic polynomial of the coefficient matrix A in (10.6.7) is

$$\begin{vmatrix} -14 - \lambda & 39 \\ -6 & 16 - \lambda \end{vmatrix} = (\lambda - 1)^2 + 9.$$

Hence, $\lambda = 1 + 3i$ is an eigenvalue of A . The associated eigenvectors satisfy $(A - (1 + 3i)I)\mathbf{x} = \mathbf{0}$. The augmented matrix of this system is

$$\begin{bmatrix} -15 - 3i & 39 & \vdots & 0 \\ -6 & 15 - 3i & \vdots & 0 \end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix} 1 & \frac{-5+i}{2} & \vdots & 0 \\ 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Therefore $x_1 = (5 - i)x_2/2$. Taking $x_2 = 2$ yields $x_1 = 5 - i$, so

$$\mathbf{x} = \begin{bmatrix} 5 - i \\ 2 \end{bmatrix}$$

is an eigenvector. The real and imaginary parts of

$$e^t(\cos 3t + i \sin 3t) \begin{bmatrix} 5 - i \\ 2 \end{bmatrix}$$

are

$$\mathbf{y}_1 = e^t \begin{bmatrix} \sin 3t + 5 \cos 3t \\ 2 \cos 3t \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = e^t \begin{bmatrix} -\cos 3t + 5 \sin 3t \\ 2 \sin 3t \end{bmatrix},$$

which are linearly independent solutions of (10.6.7). The general solution of (10.6.7) is

$$\mathbf{y} = c_1 e^t \begin{bmatrix} \sin 3t + 5 \cos 3t \\ 2 \cos 3t \end{bmatrix} + c_2 e^t \begin{bmatrix} -\cos 3t + 5 \sin 3t \\ 2 \sin 3t \end{bmatrix}.$$

Example 10.6.3 Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} -5 & 5 & 4 \\ -8 & 7 & 6 \\ 1 & 0 & 0 \end{bmatrix} \mathbf{y}. \quad (10.6.8)$$

Solution The characteristic polynomial of the coefficient matrix A in (10.6.8) is

$$\begin{vmatrix} -5 - \lambda & 5 & 4 \\ -8 & 7 - \lambda & 6 \\ 1 & 0 & -\lambda \end{vmatrix} = -(\lambda - 2)(\lambda^2 + 1).$$

Hence, the eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = i$, and $\lambda_3 = -i$. The augmented matrix of $(A - 2I)\mathbf{x} = \mathbf{0}$ is

$$\left[\begin{array}{ccc|c} -7 & 5 & 4 & 0 \\ -8 & 5 & 6 & 0 \\ 1 & 0 & -2 & 0 \end{array} \right],$$

which is row equivalent to

$$\left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

Therefore $x_1 = x_2 = 2x_3$. Taking $x_3 = 1$ yields

$$\mathbf{x}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix},$$

so

$$\mathbf{y}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} e^{2t}$$

is a solution of (10.6.8).

The augmented matrix of $(A - iI)\mathbf{x} = \mathbf{0}$ is

$$\left[\begin{array}{ccc|c} -5 - i & 5 & 4 & 0 \\ -8 & 7 - i & 6 & 0 \\ 1 & 0 & -i & 0 \end{array} \right],$$

which is row equivalent to

$$\begin{bmatrix} 1 & 0 & -i & \vdots & 0 \\ 0 & 1 & 1-i & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Therefore $x_1 = ix_3$ and $x_2 = -(1-i)x_3$. Taking $x_3 = 1$ yields the eigenvector

$$\mathbf{x}_2 = \begin{bmatrix} i \\ -1+i \\ 1 \end{bmatrix}.$$

The real and imaginary parts of

$$(\cos t + i \sin t) \begin{bmatrix} i \\ -1+i \\ 1 \end{bmatrix}$$

are

$$\mathbf{y}_2 = \begin{bmatrix} -\sin t \\ -\cos t - \sin t \\ \cos t \end{bmatrix} \quad \text{and} \quad \mathbf{y}_3 = \begin{bmatrix} \cos t \\ \cos t - \sin t \\ \sin t \end{bmatrix},$$

which are solutions of (10.6.8). Since the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ at $t = 0$ is

$$\begin{vmatrix} 2 & 0 & 1 \\ 2 & -1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 1,$$

$\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is a fundamental set of solutions of (10.6.8). The general solution of (10.6.8) is

$$\mathbf{y} = c_1 \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} -\sin t \\ -\cos t - \sin t \\ \cos t \end{bmatrix} + c_3 \begin{bmatrix} \cos t \\ \cos t - \sin t \\ \sin t \end{bmatrix}.$$

Example 10.6.4 Find the general solution of

$$\mathbf{y}' = \begin{bmatrix} 1 & -1 & -2 \\ 1 & 3 & 2 \\ 1 & -1 & 2 \end{bmatrix} \mathbf{y}. \quad (10.6.9)$$

Solution The characteristic polynomial of the coefficient matrix A in (10.6.9) is

$$\begin{vmatrix} 1-\lambda & -1 & -2 \\ 1 & 3-\lambda & 2 \\ 1 & -1 & 2-\lambda \end{vmatrix} = -(\lambda-2)((\lambda-2)^2+4).$$

Hence, the eigenvalues of A are $\lambda_1 = 2$, $\lambda_2 = 2 + 2i$, and $\lambda_3 = 2 - 2i$. The augmented matrix of $(A - 2I)\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} -1 & -1 & -2 & \vdots & 0 \\ 1 & 1 & 2 & \vdots & 0 \\ 1 & -1 & 0 & \vdots & 0 \end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix} 1 & 0 & 1 & \vdots & 0 \\ 0 & 1 & 1 & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Therefore $x_1 = x_2 = -x_3$. Taking $x_3 = 1$ yields

$$\mathbf{x}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix},$$

so

$$\mathbf{y}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{2t}$$

is a solution of (10.6.9).

The augmented matrix of $(A - (2 + 2i)I)\mathbf{x} = \mathbf{0}$ is

$$\begin{bmatrix} -1 - 2i & -1 & -2 & \vdots & 0 \\ 1 & 1 - 2i & 2 & \vdots & 0 \\ 1 & -1 & -2i & \vdots & 0 \end{bmatrix},$$

which is row equivalent to

$$\begin{bmatrix} 1 & 0 & -i & \vdots & 0 \\ 0 & 1 & i & \vdots & 0 \\ 0 & 0 & 0 & \vdots & 0 \end{bmatrix}.$$

Therefore $x_1 = ix_3$ and $x_2 = -ix_3$. Taking $x_3 = 1$ yields the eigenvector

$$\mathbf{x}_2 = \begin{bmatrix} i \\ -i \\ 1 \end{bmatrix}.$$

The real and imaginary parts of

$$e^{2t}(\cos 2t + i \sin 2t) \begin{bmatrix} i \\ -i \\ 1 \end{bmatrix}$$

are

$$\mathbf{y}_2 = e^{2t} \begin{bmatrix} -\sin 2t \\ \sin 2t \\ \cos 2t \end{bmatrix} \quad \text{and} \quad \mathbf{y}_3 = e^{2t} \begin{bmatrix} \cos 2t \\ -\cos 2t \\ \sin 2t \end{bmatrix},$$

which are solutions of (10.6.9). Since the Wronskian of $\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ at $t = 0$ is

$$\begin{vmatrix} -1 & 0 & 1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{vmatrix} = -2,$$

$\{\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3\}$ is a fundamental set of solutions of (10.6.9). The general solution of (10.6.9) is

$$\mathbf{y} = c_1 \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} e^{2t} + c_2 e^{2t} \begin{bmatrix} -\sin 2t \\ \sin 2t \\ \cos 2t \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} \cos 2t \\ -\cos 2t \\ \sin 2t \end{bmatrix}.$$

Example 10.6.5 In Example 10.6.1 we found the eigenvector

$$\mathbf{x} = \begin{bmatrix} 3 + 4i \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} + i \begin{bmatrix} 4 \\ 0 \end{bmatrix}$$

for the matrix of the system (10.6.6). Here $\mathbf{u} = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 4 \\ 0 \end{bmatrix}$ are not orthogonal, since $(\mathbf{u}, \mathbf{v}) = 12$. Since $\|\mathbf{v}\|^2 - \|\mathbf{u}\|^2 = -18$, (10.6.12) is equivalent to

$$2k^2 - 3k - 2 = 0.$$

The zeros of this equation are $k_1 = 2$ and $k_2 = -1/2$. Letting $k = 2$ in (10.6.11) yields

$$\mathbf{u}_1 = \mathbf{u} - 2\mathbf{v} = \begin{bmatrix} -5 \\ 5 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_1 = \mathbf{v} + 2\mathbf{u} = \begin{bmatrix} 10 \\ 10 \end{bmatrix},$$

and $(\mathbf{u}_1, \mathbf{v}_1) = 0$. Letting $k = -1/2$ in (10.6.11) yields

$$\mathbf{u}_1 = \mathbf{u} + \frac{\mathbf{v}}{2} = \begin{bmatrix} 5 \\ 5 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_1 = \mathbf{v} - \frac{\mathbf{u}}{2} = \frac{1}{2} \begin{bmatrix} -5 \\ 5 \end{bmatrix},$$

and again $(\mathbf{u}_1, \mathbf{v}_1) = 0$. ■

Section 10.7

Example 10.7.1

(a) Find a particular solution of the system

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 2e^{4t} \\ e^{4t} \end{bmatrix}, \quad (10.7.3)$$

which we considered in Example 10.2.1.

(b) Find the general solution of (10.7.3).

Solution (a) The complementary system is

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{y}. \quad (10.7.4)$$

The characteristic polynomial of the coefficient matrix is

$$\begin{vmatrix} 1 - \lambda & 2 \\ 2 & 1 - \lambda \end{vmatrix} = (\lambda + 1)(\lambda - 3).$$

Using the method of Section 10.4, we find that

$$\mathbf{y}_1 = \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} \quad \text{and} \quad \mathbf{y}_2 = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}$$

are linearly independent solutions of (10.7.4). Therefore

$$Y = \begin{bmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{bmatrix}$$

is a fundamental matrix for (10.7.4). We seek a particular solution $\mathbf{y}_p = Y\mathbf{u}$ of (10.7.3), where $Y\mathbf{u}' = \mathbf{f}$; that is,

$$\begin{bmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 2e^{4t} \\ e^{4t} \end{bmatrix}.$$

The determinant of Y is the Wronskian

$$\begin{vmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{vmatrix} = -2e^{2t}.$$

By Cramer's rule,

$$\begin{aligned} u_1' &= -\frac{1}{2e^{2t}} \begin{vmatrix} 2e^{4t} & e^{-t} \\ e^{4t} & -e^{-t} \end{vmatrix} = \frac{3e^{3t}}{2e^{2t}} = \frac{3}{2}e^t, \\ u_2' &= -\frac{1}{2e^{2t}} \begin{vmatrix} e^{3t} & 2e^{4t} \\ e^{3t} & e^{4t} \end{vmatrix} = \frac{e^{7t}}{2e^{2t}} = \frac{1}{2}e^{5t}. \end{aligned}$$

Therefore

$$\mathbf{u}' = \frac{1}{2} \begin{bmatrix} 3e^t \\ e^{5t} \end{bmatrix}.$$

Integrating and taking the constants of integration to be zero yields

$$\mathbf{u} = \frac{1}{10} \begin{bmatrix} 15e^t \\ e^{5t} \end{bmatrix},$$

so

$$\mathbf{y}_p = Y\mathbf{u} = \frac{1}{10} \begin{bmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{bmatrix} \begin{bmatrix} 15e^t \\ e^{5t} \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 8e^{4t} \\ 7e^{4t} \end{bmatrix}$$

is a particular solution of (10.7.3).

Solution (b) From Theorem 10.7.1, the general solution of (10.7.3) is

$$\mathbf{y} = \mathbf{y}_p + c_1\mathbf{y}_1 + c_2\mathbf{y}_2 = \frac{1}{5} \begin{bmatrix} 8e^{4t} \\ 7e^{4t} \end{bmatrix} + c_1 \begin{bmatrix} e^{3t} \\ e^{3t} \end{bmatrix} + c_2 \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}, \quad (10.7.5)$$

which can also be written as

$$\mathbf{y} = \frac{1}{5} \begin{bmatrix} 8e^{4t} \\ 7e^{4t} \end{bmatrix} + \begin{bmatrix} e^{3t} & e^{-t} \\ e^{3t} & -e^{-t} \end{bmatrix} \mathbf{c},$$

where $\mathbf{c}$ is an arbitrary constant vector.

Writing (10.7.5) in terms of coordinates yields

$$\begin{aligned} y_1 &= \frac{8}{5}e^{4t} + c_1e^{3t} + c_2e^{-t} \\ y_2 &= \frac{7}{5}e^{4t} + c_1e^{3t} - c_2e^{-t}, \end{aligned}$$

so our result is consistent with Example 10.2.1. ■

Example 10.7.2 Find a particular solution of

$$\mathbf{y}' = \begin{bmatrix} 2 & 2e^{-2t} \\ 2e^{2t} & 4 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad (10.7.6)$$

given that

$$Y = \begin{bmatrix} e^{4t} & -1 \\ e^{6t} & e^{2t} \end{bmatrix}$$

is a fundamental matrix for the complementary system.

Solution We seek a particular solution $\mathbf{y}_p = Y\mathbf{u}$ of (10.7.6) where $Y\mathbf{u}' = \mathbf{f}$; that is,

$$\begin{bmatrix} e^{4t} & -1 \\ e^{6t} & e^{2t} \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The determinant of Y is the Wronskian

$$\begin{vmatrix} e^{4t} & -1 \\ e^{6t} & e^{2t} \end{vmatrix} = 2e^{6t}.$$

By Cramer's rule,

$$\begin{aligned} u'_1 &= \frac{1}{2e^{6t}} \begin{vmatrix} 1 & -1 \\ 1 & e^{2t} \end{vmatrix} = \frac{e^{2t} + 1}{2e^{6t}} = \frac{e^{-4t} + e^{-6t}}{2} \\ u'_2 &= \frac{1}{2e^{6t}} \begin{vmatrix} e^{4t} & 1 \\ e^{6t} & 1 \end{vmatrix} = \frac{e^{4t} - e^{6t}}{2e^{6t}} = \frac{e^{-2t} - 1}{2}. \end{aligned}$$

Therefore

$$\mathbf{u}' = \frac{1}{2} \begin{bmatrix} e^{-4t} + e^{-6t} \\ e^{-2t} - 1 \end{bmatrix}.$$

Integrating and taking the constants of integration to be zero yields

$$\mathbf{u} = -\frac{1}{24} \begin{bmatrix} 3e^{-4t} + 2e^{-6t} \\ 6e^{-2t} + 12t \end{bmatrix},$$

so

$$\mathbf{y}_p = Y\mathbf{u} = -\frac{1}{24} \begin{bmatrix} e^{4t} & -1 \\ e^{6t} & e^{2t} \end{bmatrix} \begin{bmatrix} 3e^{-4t} + 2e^{-6t} \\ 6e^{-2t} + 12t \end{bmatrix} = \frac{1}{24} \begin{bmatrix} 4e^{-2t} + 12t - 3 \\ -3e^{2t}(4t + 1) - 8 \end{bmatrix}$$

is a particular solution of (10.7.6).

Example 10.7.3 Find a particular solution of

$$\mathbf{y}' = -\frac{2}{t^2} \begin{bmatrix} t & -3t^2 \\ 1 & -2t \end{bmatrix} \mathbf{y} + t^2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad (10.7.7)$$

given that

$$Y = \begin{bmatrix} 2t & 3t^2 \\ 1 & 2t \end{bmatrix}$$

is a fundamental matrix for the complementary system on $(-\infty, 0)$ and $(0, \infty)$.

Solution We seek a particular solution $\mathbf{y}_p = Y\mathbf{u}$ of (10.7.7) where $Y\mathbf{u}' = \mathbf{f}$; that is,

$$\begin{bmatrix} 2t & 3t^2 \\ 1 & 2t \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \end{bmatrix} = \begin{bmatrix} t^2 \\ t^2 \end{bmatrix}.$$

The determinant of Y is the Wronskian

$$\begin{vmatrix} 2t & 3t^2 \\ 1 & 2t \end{vmatrix} = t^2.$$

By Cramer's rule,

$$\begin{aligned} u'_1 &= \frac{1}{t^2} \begin{vmatrix} t^2 & 3t^2 \\ t^2 & 2t \end{vmatrix} = \frac{2t^3 - 3t^4}{t^2} = 2t - 3t^2, \\ u'_2 &= \frac{1}{t^2} \begin{vmatrix} 2t & t^2 \\ 1 & t^2 \end{vmatrix} = \frac{2t^3 - t^2}{t^2} = 2t - 1. \end{aligned}$$

Therefore

$$\mathbf{u}' = \begin{bmatrix} 2t - 3t^2 \\ 2t - 1 \end{bmatrix}.$$

Integrating and taking the constants of integration to be zero yields

$$\mathbf{u} = \begin{bmatrix} t^2 - t^3 \\ t^2 - t \end{bmatrix},$$

so

$$\mathbf{y}_p = Y\mathbf{u} = \begin{bmatrix} 2t & 3t^2 \\ 1 & 2t \end{bmatrix} \begin{bmatrix} t^2 - t^3 \\ t^2 - t \end{bmatrix} = \begin{bmatrix} t^3(t-1) \\ t^2(t-1) \end{bmatrix}$$

is a particular solution of (10.7.7).

Example 10.7.4

(a) Find a particular solution of

$$\mathbf{y}' = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \mathbf{y} + \begin{bmatrix} e^t \\ 0 \\ e^{-t} \end{bmatrix}. \quad (10.7.8)$$

(b) Find the general solution of (10.7.8).

Solution (a) The complementary system for (10.7.8) is

$$\mathbf{y}' = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{bmatrix} \mathbf{y}. \quad (10.7.9)$$

The characteristic polynomial of the coefficient matrix is

$$\begin{vmatrix} 2-\lambda & -1 & -1 \\ 1 & -\lambda & -1 \\ 1 & -1 & -\lambda \end{vmatrix} = -\lambda(\lambda-1)^2.$$

Using the method of Section 10.4, we find that

$$\mathbf{y}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{y}_2 = \begin{bmatrix} e^t \\ e^t \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{y}_3 = \begin{bmatrix} e^t \\ 0 \\ e^t \end{bmatrix}$$

are linearly independent solutions of (10.7.9). Therefore

$$Y = \begin{bmatrix} 1 & e^t & e^t \\ 1 & e^t & 0 \\ 1 & 0 & e^t \end{bmatrix}$$

is a fundamental matrix for (10.7.9). We seek a particular solution $\mathbf{y}_p = Y\mathbf{u}$ of (10.7.8), where $Y\mathbf{u}' = \mathbf{f}$; that is,

$$\begin{bmatrix} 1 & e^t & e^t \\ 1 & e^t & 0 \\ 1 & 0 & e^t \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \\ u'_3 \end{bmatrix} = \begin{bmatrix} e^t \\ 0 \\ e^{-t} \end{bmatrix}.$$

The determinant of Y is the Wronskian

$$\begin{vmatrix} 1 & e^t & e^t \\ 1 & e^t & 0 \\ 1 & 0 & e^t \end{vmatrix} = -e^{2t}.$$

Thus, by Cramer's rule,

$$\begin{aligned} u'_1 &= -\frac{1}{e^{2t}} \begin{vmatrix} e^t & e^t & e^t \\ 0 & e^t & 0 \\ e^{-t} & 0 & e^t \end{vmatrix} = -\frac{e^{3t} - e^t}{e^{2t}} = e^{-t} - e^t \\ u'_2 &= -\frac{1}{e^{2t}} \begin{vmatrix} 1 & e^t & e^t \\ 1 & 0 & 0 \\ 1 & e^{-t} & e^t \end{vmatrix} = -\frac{1 - e^{2t}}{e^{2t}} = 1 - e^{-2t} \\ u'_3 &= -\frac{1}{e^{2t}} \begin{vmatrix} 1 & e^t & e^t \\ 1 & e^t & 0 \\ 1 & 0 & e^{-t} \end{vmatrix} = \frac{e^{2t}}{e^{2t}} = 1. \end{aligned}$$

Therefore

$$\mathbf{u}' = \begin{bmatrix} e^{-t} - e^t \\ 1 - e^{-2t} \\ 1 \end{bmatrix}.$$

Integrating and taking the constants of integration to be zero yields

$$\mathbf{u} = \begin{bmatrix} -e^t - e^{-t} \\ \frac{e^{-2t}}{2} + t \\ t \end{bmatrix},$$

so

$$\mathbf{y}_p = Y\mathbf{u} = \begin{bmatrix} 1 & e^t & e^t \\ 1 & e^t & 0 \\ 1 & 0 & e^t \end{bmatrix} \begin{bmatrix} -e^t - e^{-t} \\ \frac{e^{-2t}}{2} + t \\ t \end{bmatrix} = \begin{bmatrix} e^t(2t - 1) - \frac{e^{-t}}{2} \\ e^t(t - 1) - \frac{e^{-t}}{2} \\ e^t(t - 1) - e^{-t} \end{bmatrix}$$

is a particular solution of (10.7.8).

Solution (b) From Theorem 10.7.1 the general solution of (10.7.8) is

$$\mathbf{y} = \mathbf{y}_p + c_1 \mathbf{y}_1 + c_2 \mathbf{y}_2 + c_3 \mathbf{y}_3 = \begin{bmatrix} e^t(2t-1) - \frac{e^{-t}}{2} \\ e^t(t-1) - \frac{e^{-t}}{2} \\ e^t(t-1) - e^{-t} \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} e^t \\ e^t \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} e^t \\ 0 \\ e^t \end{bmatrix},$$

which can be written as

$$\mathbf{y} = \mathbf{y}_p + Y\mathbf{c} = \begin{bmatrix} e^t(2t-1) - \frac{e^{-t}}{2} \\ e^t(t-1) - \frac{e^{-t}}{2} \\ e^t(t-1) - e^{-t} \end{bmatrix} + \begin{bmatrix} 1 & e^t & e^t \\ 1 & e^t & 0 \\ 1 & 0 & e^t \end{bmatrix} \mathbf{c}$$

where $\mathbf{c}$ is an arbitrary constant vector.

Example 10.7.5 Find a particular solution of

$$\mathbf{y}' = \frac{1}{2} \begin{bmatrix} 3 & e^{-t} & -e^{2t} \\ 0 & 6 & 0 \\ -e^{-2t} & e^{-3t} & -1 \end{bmatrix} \mathbf{y} + \begin{bmatrix} 1 \\ e^t \\ e^{-t} \end{bmatrix}, \quad (10.7.10)$$

given that

$$Y = \begin{bmatrix} e^t & 0 & e^{2t} \\ 0 & e^{3t} & e^{3t} \\ e^{-t} & 1 & 0 \end{bmatrix}$$

is a fundamental matrix for the complementary system.

Solution We seek a particular solution of (10.7.10) in the form $\mathbf{y}_p = Y\mathbf{u}$, where $Y\mathbf{u}' = \mathbf{f}$; that is,

$$\begin{bmatrix} e^t & 0 & e^{2t} \\ 0 & e^{3t} & e^{3t} \\ e^{-t} & 1 & 0 \end{bmatrix} \begin{bmatrix} u'_1 \\ u'_2 \\ u'_3 \end{bmatrix} = \begin{bmatrix} 1 \\ e^t \\ e^{-t} \end{bmatrix}.$$

The determinant of Y is the Wronskian

$$\begin{vmatrix} e^t & 0 & e^{2t} \\ 0 & e^{3t} & e^{3t} \\ e^{-t} & 1 & 0 \end{vmatrix} = -2e^{4t}.$$

By Cramer's rule,

$$\begin{aligned} u'_1 &= -\frac{1}{2e^{4t}} \begin{vmatrix} 1 & 0 & e^{2t} \\ e^t & e^{3t} & e^{3t} \\ e^{-t} & 1 & 0 \end{vmatrix} = \frac{e^{4t}}{2e^{4t}} = \frac{1}{2} \\ u'_2 &= -\frac{1}{2e^{4t}} \begin{vmatrix} e^t & 1 & e^{2t} \\ 0 & e^t & e^{3t} \\ e^{-t} & e^{-t} & 0 \end{vmatrix} = \frac{e^{3t}}{2e^{4t}} = \frac{1}{2}e^{-t} \\ u'_3 &= -\frac{1}{2e^{4t}} \begin{vmatrix} e^t & 0 & 1 \\ 0 & e^{3t} & e^t \\ e^{-t} & 1 & e^{-t} \end{vmatrix} = -\frac{e^{3t} - 2e^{2t}}{2e^{4t}} = \frac{2e^{-2t} - e^{-t}}{2}. \end{aligned}$$

Therefore

$$\mathbf{u}' = \frac{1}{2} \begin{bmatrix} 1 \\ e^{-t} \\ 2e^{-2t} - e^{-t} \end{bmatrix}.$$

Integrating and taking the constants of integration to be zero yields

$$\mathbf{u} = \frac{1}{2} \begin{bmatrix} t \\ -e^{-t} \\ e^{-t} - e^{-2t} \end{bmatrix},$$

so

$$\mathbf{y}_p = Y\mathbf{u} = \frac{1}{2} \begin{bmatrix} e^t & 0 & e^{2t} \\ 0 & e^{3t} & e^{3t} \\ e^{-t} & 1 & 0 \end{bmatrix} \begin{bmatrix} t \\ -e^{-t} \\ e^{-t} - e^{-2t} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} e^t(t+1) - 1 \\ -e^t \\ e^{-t}(t-1) \end{bmatrix}$$

is a particular solution of (10.7.10).

Section 11.1

Example 11.1.1 (Problem 1) Solve the eigenvalue problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y(L) = 0. \quad (11.1.3)$$

Solution From Theorem 11.1.1, any eigenvalues of (11.1.3) must be positive. If y satisfies (11.1.3) with $\lambda > 0$, then

$$y = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x,$$

where c_1 and c_2 are constants. The boundary condition $y(0) = 0$ implies that $c_1 = 0$. Therefore $y = c_2 \sin \sqrt{\lambda} x$. Now the boundary condition $y(L) = 0$ implies that $c_2 \sin \sqrt{\lambda} L = 0$. To make $c_2 \sin \sqrt{\lambda} L = 0$ with $c_2 \neq 0$, we must choose $\sqrt{\lambda} = n\pi/L$, where n is a positive integer. Therefore $\lambda_n = n^2\pi^2/L^2$ is an eigenvalue and

$$y_n = \sin \frac{n\pi x}{L}$$

is an associated eigenfunction. ■

Example 11.1.2 (Problem 3) Solve the eigenvalue problem

$$y'' + \lambda y = 0, \quad y(0) = 0, \quad y'(L) = 0. \quad (11.1.4)$$

Solution From Theorem 11.1.1, any eigenvalues of (11.1.4) must be positive. If y satisfies (11.1.4) with $\lambda > 0$, then

$$y = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x,$$

where c_1 and c_2 are constants. The boundary condition $y(0) = 0$ implies that $c_1 = 0$. Therefore $y = c_2 \sin \sqrt{\lambda} x$. Hence, $y' = c_2 \sqrt{\lambda} \cos \sqrt{\lambda} x$ and the boundary condition $y'(L) = 0$ implies that $c_2 \cos \sqrt{\lambda} L = 0$. To make $c_2 \cos \sqrt{\lambda} L = 0$ with $c_2 \neq 0$ we must choose

$$\sqrt{\lambda} = \frac{(2n-1)\pi}{2L},$$

where n is a positive integer. Then $\lambda_n = (2n-1)^2\pi^2/4L^2$ is an eigenvalue and

$$y_n = \sin \frac{(2n-1)\pi x}{2L}$$

is an associated eigenfunction. ■

Example 11.1.3 (Problem 5) Solve the eigenvalue problem

$$y'' + \lambda y = 0, \quad y(-L) = y(L), \quad y'(-L) = y'(L). \quad (11.1.5)$$

Solution From Theorem 11.1.1, $\lambda = 0$ is an eigenvalue of (11.1.5) with associated eigenfunction $y_0 = 1$, and any other eigenvalues must be positive. If y satisfies (11.1.5) with $\lambda > 0$, then

$$y = c_1 \cos \sqrt{\lambda} x + c_2 \sin \sqrt{\lambda} x, \quad (11.1.6)$$

where c_1 and c_2 are constants. The boundary condition $y(-L) = y(L)$ implies that

$$c_1 \cos(-\sqrt{\lambda} L) + c_2 \sin(-\sqrt{\lambda} L) = c_1 \cos \sqrt{\lambda} L + c_2 \sin \sqrt{\lambda} L. \quad (11.1.7)$$

Since

$$\cos(-\sqrt{\lambda} L) = \cos \sqrt{\lambda} L \quad \text{and} \quad \sin(-\sqrt{\lambda} L) = -\sin \sqrt{\lambda} L, \quad (11.1.8)$$

(11.1.7) implies that

$$c_2 \sin \sqrt{\lambda} L = 0. \quad (11.1.9)$$

Differentiating (11.1.6) yields

$$y' = \sqrt{\lambda} (-c_1 \sin \sqrt{\lambda} x + c_2 \cos \sqrt{\lambda} x).$$

The boundary condition $y'(-L) = y'(L)$ implies that

$$-c_1 \sin(-\sqrt{\lambda} L) + c_2 \cos(-\sqrt{\lambda} L) = -c_1 \sin \sqrt{\lambda} L + c_2 \cos \sqrt{\lambda} L,$$

and (11.1.8) implies that

$$c_1 \sin \sqrt{\lambda} L = 0. \quad (11.1.10)$$

Eqns. (11.1.9) and (11.1.10) imply that $c_1 = c_2 = 0$ unless $\sqrt{\lambda} = n\pi/L$, where n is a positive integer. In this case (11.1.9) and (11.1.10) both hold for arbitrary c_1 and c_2 . The eigenvalue determined in this way is $\lambda_n = n^2\pi^2/L^2$, and each such eigenvalue has the linearly independent associated eigenfunctions

$$\cos \frac{n\pi x}{L} \quad \text{and} \quad \sin \frac{n\pi x}{L}. \quad \blacksquare$$

Example 11.1.4 Show that the eigenfunctions

$$1, \cos \frac{\pi x}{L}, \sin \frac{\pi x}{L}, \cos \frac{2\pi x}{L}, \sin \frac{2\pi x}{L}, \dots, \cos \frac{n\pi x}{L}, \sin \frac{n\pi x}{L}, \dots \quad (11.1.11)$$

of Problem 5 are orthogonal on $[-L, L]$.

Solution We must show that

$$\int_{-L}^L f(x)g(x) dx = 0 \quad (11.1.12)$$

whenever f and g are distinct functions from (11.1.11). If r is any nonzero integer, then

$$\int_{-L}^L \cos \frac{r\pi x}{L} dx = \frac{L}{r\pi} \sin \frac{r\pi x}{L} \Big|_{-L}^L = 0. \quad (11.1.13)$$

and

$$\int_{-L}^L \sin \frac{r\pi x}{L} dx = -\frac{L}{r\pi} \cos \frac{r\pi x}{L} \Big|_{-L}^L = 0.$$

Therefore (11.1.12) holds if $f \equiv 1$ and g is any other function in (11.1.11).

If $f(x) = \cos m\pi x/L$ and $g(x) = \cos n\pi x/L$ where m and n are distinct positive integers, then

$$\int_{-L}^L f(x)g(x) dx = \int_{-L}^L \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L} dx. \quad (11.1.14)$$

To evaluate this integral, we use the identity

$$\cos A \cos B = \frac{1}{2} [\cos(A - B) + \cos(A + B)]$$

with $A = m\pi x/L$ and $B = n\pi x/L$. Then (11.1.14) becomes

$$\int_{-L}^L f(x)g(x) dx = \frac{1}{2} \left[\int_{-L}^L \cos \frac{(m-n)\pi x}{L} dx + \int_{-L}^L \cos \frac{(m+n)\pi x}{L} dx \right].$$

Since $m - n$ and $m + n$ are both nonzero integers, (11.1.13) implies that the integrals on the right are both zero. Therefore (11.1.12) is true in this case.

If $f(x) = \sin m\pi x/L$ and $g(x) = \sin n\pi x/L$ where m and n are distinct positive integers, then

$$\int_{-L}^L f(x)g(x) dx = \int_{-L}^L \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} dx. \quad (11.1.15)$$

To evaluate this integral, we use the identity

$$\sin A \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)]$$

with $A = m\pi x/L$ and $B = n\pi x/L$. Then (11.1.15) becomes

$$\int_{-L}^L f(x)g(x) dx = \frac{1}{2} \left[\int_{-L}^L \cos \frac{(m-n)\pi x}{L} dx - \int_{-L}^L \cos \frac{(m+n)\pi x}{L} dx \right] = 0.$$

If $f(x) = \sin m\pi x/L$ and $g(x) = \cos n\pi x/L$ where m and n are positive integers (not necessarily distinct), then

$$\int_{-L}^L f(x)g(x) dx = \int_{-L}^L \sin \frac{m\pi x}{L} \cos \frac{n\pi x}{L} dx = 0$$

because the integrand is an odd function and the limits are symmetric about $x = 0$. ■

Section 11.2

Example 11.2.1 Find the Fourier series of the piecewise smooth function

$$f(x) = \begin{cases} -x, & -2 < x < 0, \\ \frac{1}{2}, & 0 < x < 2 \end{cases}$$

on $[-2, 2]$ (Figure 11.2.1). Determine the sum of the Fourier series for $-2 \leq x \leq 2$.

Solution Note that we haven't bothered to define $f(-2)$, $f(0)$, and $f(2)$. No matter how they may be defined, f is piecewise smooth on $[-2, 2]$, and the coefficients in the Fourier series

$$F(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{2} + b_n \sin \frac{n\pi x}{2} \right)$$

are not affected by them. In any case, Theorem 11.2.4 implies that $F(x) = f(x)$ in $(-2, 0)$ and $(0, 2)$, where f is continuous, while

$$F(-2) = F(2) = \frac{f(-2+) + f(2-)}{2} = \frac{1}{2} \left(2 + \frac{1}{2} \right) = \frac{5}{4}$$

and

$$F(0) = \frac{f(0-) + f(0+)}{2} = \frac{1}{2} \left(0 + \frac{1}{2} \right) = \frac{1}{4}.$$

To summarize,

$$F(x) = \begin{cases} \frac{5}{4}, & x = -2 \\ -x, & -2 < x < 0, \\ \frac{1}{4}, & x = 0, \\ \frac{1}{2}, & 0 < x < 2, \\ \frac{5}{4}, & x = 2. \end{cases}$$

We compute the Fourier coefficients as follows:

$$a_0 = \frac{1}{4} \int_{-2}^2 f(x) dx = \frac{1}{4} \left[\int_{-2}^0 (-x) dx + \int_0^2 \frac{1}{2} dx \right] = \frac{3}{4}.$$

If $n \geq 1$, then

$$\begin{aligned} a_n &= \frac{1}{2} \int_{-2}^2 f(x) \cos \frac{n\pi x}{2} dx = \frac{1}{2} \left[\int_{-2}^0 (-x) \cos \frac{n\pi x}{2} dx + \int_0^2 \frac{1}{2} \cos \frac{n\pi x}{2} dx \right] \\ &= \frac{2}{n^2 \pi^2} (\cos n\pi - 1), \end{aligned}$$

and

$$\begin{aligned} b_n &= \frac{1}{2} \int_{-2}^2 f(x) \sin \frac{n\pi x}{2} dx = \frac{1}{2} \left[\int_{-2}^0 (-x) \sin \frac{n\pi x}{2} dx + \int_0^2 \frac{1}{2} \sin \frac{n\pi x}{2} dx \right] \\ &= \frac{1}{2n\pi} (1 + 3 \cos n\pi). \end{aligned}$$

Therefore

$$F(x) = \frac{3}{4} + \frac{2}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos n\pi - 1}{n^2} \cos \frac{n\pi x}{2} + \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{1 + 3 \cos n\pi}{n} \sin \frac{n\pi x}{2}.$$

Figure 11.2.2 shows how the partial sum

$$F_m(x) = \frac{3}{4} + \frac{2}{\pi^2} \sum_{n=1}^m \frac{\cos n\pi - 1}{n^2} \cos \frac{n\pi x}{2} + \frac{1}{2\pi} \sum_{n=1}^m \frac{1 + 3 \cos n\pi}{n} \sin \frac{n\pi x}{2}$$

approximates $f(x)$ for $m = 5$ (dotted curve), $m = 10$ (dashed curve), and $m = 15$ (solid curve).

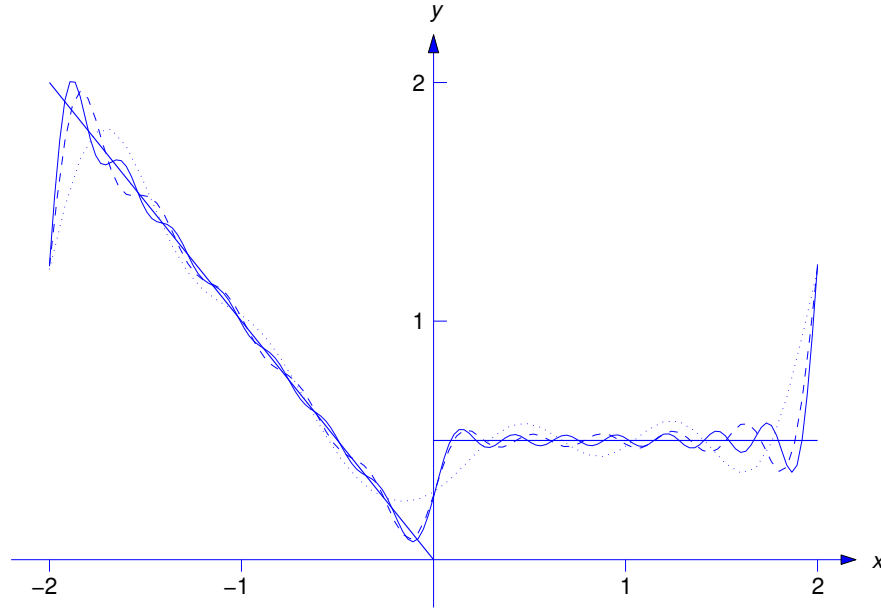


Figure 11.2.2

Example 11.2.2 The functions $u(x) = \cos \omega x$ and $u(x) = x^2$ are even, while $v(x) = \sin \omega x$ and $v(x) = x^3$ are odd. The function $w(x) = e^x$ is neither even nor odd.

Example 11.2.3 Find the Fourier series of $f(x) = x^2 - x$ on $[-2, 2]$, and determine its sum for $-2 \leq x \leq 2$.

Solution Since $L = 2$,

$$F(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{2} + b_n \sin \frac{n\pi x}{2} \right)$$

where

$$a_0 = \frac{1}{4} \int_{-2}^2 (x^2 - x) dx, \quad (11.2.9)$$

$$a_n = \frac{1}{2} \int_{-2}^2 (x^2 - x) \cos \frac{n\pi x}{2} dx, \quad n = 1, 2, 3, \dots, \quad (11.2.10)$$

and

$$b_n = \frac{1}{2} \int_{-2}^2 (x^2 - x) \sin \frac{n\pi x}{2} dx, \quad n = 1, 2, 3, \dots \quad (11.2.11)$$

We simplify the evaluation of these integrals by using Theorem 11.2.5 with $u(x) = x^2$ and $v(x) = x$; thus, from (11.2.9),

$$a_0 = \frac{1}{2} \int_0^2 x^2 dx = \frac{x^3}{6} \Big|_0^2 = \frac{4}{3}.$$

From (11.2.10),

$$\begin{aligned}
 a_n &= \int_0^2 x^2 \cos \frac{n\pi x}{2} dx = \frac{2}{n\pi} \left[x^2 \sin \frac{n\pi x}{2} \Big|_0^2 - 2 \int_0^2 x \sin \frac{n\pi x}{2} dx \right] \\
 &= \frac{8}{n^2 \pi^2} \left[x \cos \frac{n\pi x}{2} \Big|_0^2 - \int_0^2 \cos \frac{n\pi x}{2} dx \right] \\
 &= \frac{8}{n^2 \pi^2} \left[2 \cos n\pi - \frac{2}{n\pi} \sin \frac{n\pi x}{2} \Big|_0^2 \right] = (-1)^n \frac{16}{n^2 \pi^2}.
 \end{aligned}$$

From (11.2.11),

$$\begin{aligned}
 b_n &= - \int_0^2 x \sin \frac{n\pi x}{2} dx = \frac{2}{n\pi} \left[x \cos \frac{n\pi x}{2} \Big|_0^2 - \int_0^2 \cos \frac{n\pi x}{2} dx \right] \\
 &= \frac{2}{n\pi} \left[2 \cos n\pi - \frac{2}{n\pi} \sin \frac{n\pi x}{2} \Big|_0^2 \right] = (-1)^n \frac{4}{n\pi}.
 \end{aligned}$$

Therefore

$$F(x) = \frac{4}{3} + \frac{16}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos \frac{n\pi x}{2} + \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin \frac{n\pi x}{2}.$$

Theorem 11.2.4 implies that

$$F(x) = \begin{cases} 4, & x = -2, \\ x^2 - x, & -2 < x < 2, \\ 4, & x = 2. \end{cases}$$

Figure 11.2.3 shows how the partial sum

$$F_m(x) = \frac{4}{3} + \frac{16}{\pi^2} \sum_{n=1}^m \frac{(-1)^n}{n^2} \cos \frac{n\pi x}{2} + \frac{4}{\pi} \sum_{n=1}^m \frac{(-1)^n}{n} \sin \frac{n\pi x}{2}$$

approximates $f(x)$ for $m = 5$ (dotted curve), $m = 10$ (dashed curve), and $m = 15$ (solid curve). ■

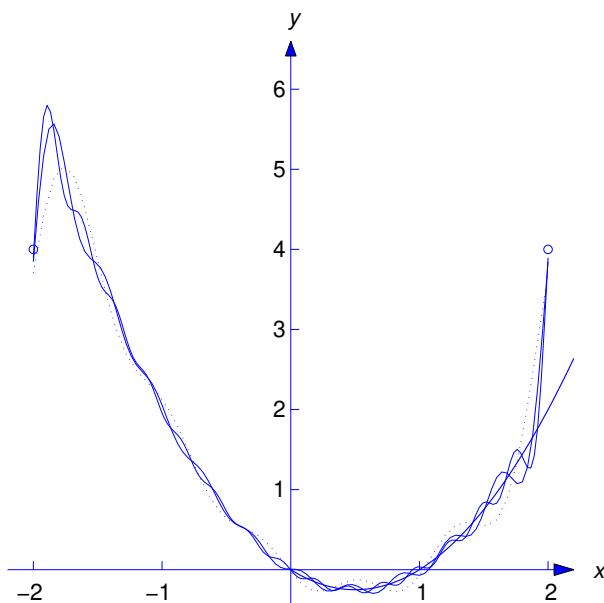


Figure 11.2.3 Approximation of $f(x) = x^2 - x$ by partial sums of its Fourier series on $[-2, 2]$

Example 11.2.4 Find the Fourier series of $f(x) = x$ on $[-\pi, \pi]$, and determine its sum for $-\pi \leq x \leq \pi$.

Solution Since f is odd and $L = \pi$,

$$F(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$\begin{aligned} b_n &= \frac{2}{\pi} \int_0^{\pi} x \sin nx \, dx = -\frac{2}{n\pi} \left[x \cos nx \Big|_0^{\pi} - \int_0^{\pi} \cos nx \, dx \right] \\ &= -\frac{2}{n} \cos n\pi + \frac{2}{n^2\pi} \sin nx \Big|_0^{\pi} = (-1)^{n+1} \frac{2}{n}. \end{aligned}$$

Therefore

$$F(x) = -2 \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin nx.$$

Theorem 11.2.4 implies that

$$F(x) = \begin{cases} 0, & x = -\pi, \\ x, & -\pi < x < \pi, \\ 0, & x = \pi. \end{cases}$$

Figure 11.2.4 shows how the partial sum

$$F_m(x) = -2 \sum_{n=1}^m \frac{(-1)^n}{n} \sin nx$$

approximates $f(x)$ for $m = 5$ (dotted curve), $m = 10$ (dashed curve), and $m = 15$ (solid curve).

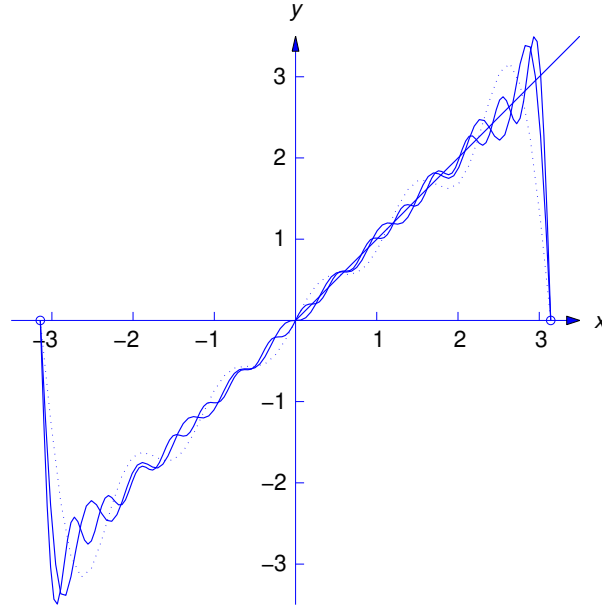


Figure 11.2.4 Approximation of $f(x) = x$ by partial sums of its Fourier series on $[-\pi, \pi]$

Example 11.2.5 Find the Fourier series of $f(x) = |x|$ on $[-\pi, \pi]$ and determine its sum for $-\pi \leq x \leq \pi$.

Solution Since f is even and $L = \pi$,

$$F(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx.$$

Since $f(x) = x$ if $x \geq 0$,

$$a_0 = \frac{1}{\pi} \int_0^{\pi} x \, dx = \frac{x^2}{2\pi} \Big|_0^{\pi} = \frac{\pi}{2}$$

and, if $n \geq 1$,

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} x \cos nx \, dx = \frac{2}{n\pi} \left[x \sin nx \Big|_0^{\pi} - \int_0^{\pi} \sin nx \, dx \right] \\ &= \frac{2}{n^2\pi} \cos nx \Big|_0^{\pi} = \frac{2}{n^2\pi} (\cos n\pi - 1) = \frac{2}{n^2\pi} [(-1)^n - 1]. \end{aligned}$$

Therefore

$$F(x) = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n - 1}{n^2} \cos nx. \quad (11.2.12)$$

However, since

$$(-1)^n - 1 = \begin{cases} 0 & \text{if } n = 2m, \\ -2 & \text{if } n = 2m + 1, \end{cases}$$

the terms in (11.2.12) for which $n = 2m$ are all zeros. Therefore we only need to include the terms for which $n = 2m + 1$; that is, we can rewrite (11.2.12) as

$$F(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{m=0}^{\infty} \frac{1}{(2m+1)^2} \cos(2m+1)x.$$

However, since the name of the index of summation doesn't matter, we prefer to replace m by n , and write

$$F(x) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} \cos(2n+1)x.$$

Since $|x|$ is continuous for all x and $|- \pi| = |\pi|$, Theorem 11.2.4 implies that $F(x) = |x|$ for all x in $[-\pi, \pi]$.

Example 11.2.6 Find the Fourier series of $f(x) = x(x^2 - L^2)$ on $[-L, L]$, and determine its sum for $-L \leq x \leq L$.

Solution Since f is odd,

$$F(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L},$$

where

$$\begin{aligned} b_n &= \frac{2}{L} \int_0^L x(x^2 - L^2) \sin \frac{n\pi x}{L} dx \\ &= -\frac{2}{n\pi} \left[x(x^2 - L^2) \cos \frac{n\pi x}{L} \Big|_0^L - \int_0^L (3x^2 - L^2) \cos \frac{n\pi x}{L} dx \right] \\ &= \frac{2L}{n^2\pi^2} \left[(3x^2 - L^2) \sin \frac{n\pi x}{L} \Big|_0^L - 6 \int_0^L x \sin \frac{n\pi x}{L} dx \right] \\ &= \frac{12L^2}{n^3\pi^3} \left[x \cos \frac{n\pi x}{L} \Big|_0^L - \int_0^L \cos \frac{n\pi x}{L} dx \right] = (-1)^n \frac{12L^3}{n^3\pi^3}. \end{aligned}$$

Therefore

$$F(x) = \frac{12L^3}{\pi^3} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin \frac{n\pi x}{L}.$$

Theorem 11.2.4 implies that $F(x) = x(x^2 - L^2)$ for all x in $[-L, L]$.

Example 11.2.7 (Gibbs Phenomenon) The Fourier series of

$$f(x) = \begin{cases} 0, & -1 < x < -\frac{1}{2}, \\ 1, & -\frac{1}{2} < x < \frac{1}{2}, \\ 0, & \frac{1}{2} < x < 1 \end{cases}$$

on $[-1, 1]$ is

$$F(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \cos(2n-1)\pi x.$$

(Verify.) According to Theorem 11.2.4,

$$F(x) = \begin{cases} 0, & -1 \leq x < -\frac{1}{2}, \\ \frac{1}{2}, & x = -\frac{1}{2}, \\ 1, & -\frac{1}{2} < x < \frac{1}{2}, \\ \frac{1}{2}, & x = \frac{1}{2}, \\ 0, & \frac{1}{2} < x \leq 1; \end{cases}$$

thus, F (as well as f) has unit jump discontinuities at $x = \pm\frac{1}{2}$. Figures 11.2.5-11.2.7 show the graphs of $y = f(x)$ and

$$y = F_{2N-1}(x) = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^N \frac{(-1)^{n-1}}{2n-1} \cos(2n-1)\pi x$$

for $N = 10, 20$, and 30 . You can see that although F_{2N-1} approximates F (and therefore f) well on larger intervals as N increases, the maximum absolute values of the errors remain approximately equal to $.09$, but occur closer to the discontinuities $x = \pm\frac{1}{2}$ as N increases.

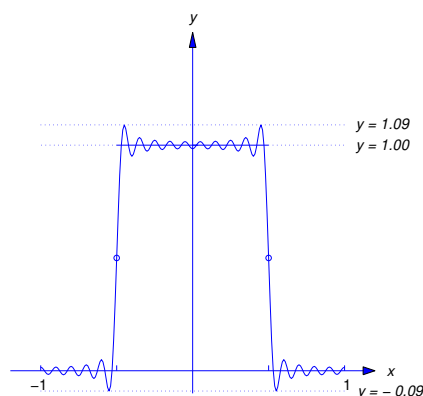


Figure 11.2.5 The Gibbs Phenomenon:
Example 11.2.7, $N = 10$

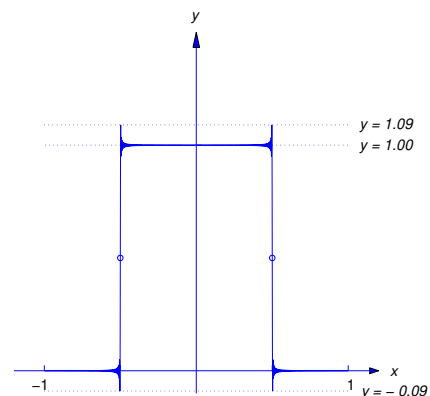


Figure 11.2.6 The Gibbs Phenomenon:
Example 11.2.7, $N = 20$

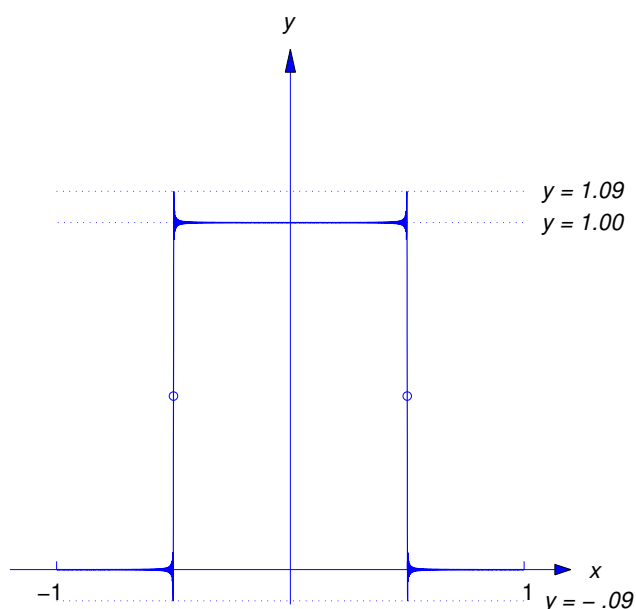


Figure 11.2.7 The Gibbs Phenomenon: Example 11.2.7, $N = 30$

Section 11.3

Example 11.3.1 Find the Fourier cosine series of $f(x) = x$ on $[0, L]$.

Solution The coefficients are

$$a_0 = \frac{1}{L} \int_0^L x \, dx = \frac{1}{L} \frac{x^2}{2} \Big|_0^L = \frac{L}{2}$$

and, if $n \geq 1$

$$\begin{aligned} a_n &= \frac{2}{L} \int_0^L x \cos \frac{n\pi x}{L} \, dx = \frac{2}{n\pi} \left[x \sin \frac{n\pi x}{L} \Big|_0^L - \int_0^L \sin \frac{n\pi x}{L} \, dx \right] \\ &= -\frac{2}{n\pi} \int_0^L \sin \frac{n\pi x}{L} \, dx = \frac{2L}{n^2\pi^2} \cos \frac{n\pi x}{L} \Big|_0^L = \frac{2L}{n^2\pi^2} [(-1)^n - 1] \\ &= \begin{cases} -\frac{4L}{(2m-1)^2\pi^2} & \text{if } n = 2m-1, \\ 0 & \text{if } n = 2m. \end{cases} \end{aligned}$$

Therefore

$$C(x) = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{L}.$$

Theorem 11.3.1 implies that

$$C(x) = x, \quad 0 \leq x \leq L.$$

Example 11.3.2 Find the Fourier sine series of $f(x) = x$ on $[0, L]$.

Solution The coefficients are

$$\begin{aligned} b_n &= \frac{2}{L} \int_0^L x \sin \frac{n\pi x}{L} \, dx = -\frac{2}{n\pi} \left[x \cos \frac{n\pi x}{L} \Big|_0^L - \int_0^L \cos \frac{n\pi x}{L} \, dx \right] \\ &= (-1)^{n+1} \frac{2L}{n\pi} + \frac{2L}{n^2\pi^2} \sin \frac{n\pi x}{L} \Big|_0^L = (-1)^{n+1} \frac{2L}{n\pi}. \end{aligned}$$

Therefore

$$S(x) = -\frac{2L}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin \frac{n\pi x}{L}.$$

Theorem 11.3.2 implies that

$$S(x) = \begin{cases} x, & 0 \leq x < L, \\ 0, & x = L. \end{cases}$$

Example 11.3.3 Find the mixed Fourier cosine series of $f(x) = x - L$ on $[0, L]$.

Solution The coefficients are

$$\begin{aligned} c_n &= \frac{2}{L} \int_0^L (x - L) \cos \frac{(2n-1)\pi x}{2L} \, dx \\ &= \frac{4}{(2n-1)\pi} \left[(x - L) \sin \frac{(2n-1)\pi x}{2L} \Big|_0^L - \int_0^L \sin \frac{(2n-1)\pi x}{2L} \, dx \right] \\ &= \frac{8L}{(2n-1)^2\pi^2} \cos \frac{(2n-1)\pi x}{2L} \Big|_0^L = -\frac{8L}{(2n-1)^2\pi^2}. \end{aligned}$$

Therefore

$$C_M(x) = -\frac{8L}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{2L}.$$

Theorem 11.3.3 implies that

$$C_M(x) = x - L, \quad 0 \leq x \leq L.$$

Example 11.3.4 Find the mixed Fourier sine series of $f(x) = x$ on $[0, L]$.

Solution The coefficients are

$$\begin{aligned} d_n &= \frac{2}{L} \int_0^L x \sin \frac{(2n-1)\pi x}{2L} dx \\ &= -\frac{4}{(2n-1)\pi} \left[x \cos \frac{(2n-1)\pi x}{2L} \Big|_0^L - \int_0^L \cos \frac{(2n-1)\pi x}{2L} dx \right] \\ &= \frac{4}{(2n-1)\pi} \int_0^L \cos \frac{(2n-1)\pi x}{2L} dx \\ &= \frac{8L}{(2n-1)^2\pi^2} \sin \frac{(2n-1)\pi x}{2L} \Big|_0^L = (-1)^{n+1} \frac{8L}{(2n-1)^2\pi^2}. \end{aligned}$$

Therefore

$$S_M(x) = -\frac{8L}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{2L}.$$

Theorem 11.3.4 implies that

$$S_M(x) = x, \quad 0 \leq x \leq L.$$

Example 11.3.5 Find the Fourier cosine expansion of $f(x) = x^2(3L - 2x)$ on $[0, L]$.

Solution Here

$$a_0 = \frac{1}{L} \int_0^L (3Lx^2 - 2x^3) dx = \frac{1}{L} \left(Lx^3 - \frac{x^4}{2} \right) \Big|_0^L = \frac{L^3}{2}$$

and

$$a_n = \frac{2}{L} \int_0^L (3Lx^2 - 2x^3) \cos \frac{n\pi x}{L} dx, \quad n \geq 1.$$

Evaluating this integral directly is laborious. However, since $f'(x) = 6Lx - 6x^2$, we see that $f'(0) = f'(L) = 0$. Since $f'''(x) = -12$, we see from (11.3.4) that if $n \geq 1$ then

$$\begin{aligned} a_n &= -\frac{24L^2}{n^3\pi^3} \int_0^L \sin \frac{n\pi x}{L} dx = \frac{24L^3}{n^4\pi^4} \cos \frac{n\pi x}{L} \Big|_0^L = \frac{24L^3}{n^4\pi^4} [(-1)^n - 1] \\ &= \begin{cases} -\frac{48L^3}{(2m-1)^4\pi^4} & \text{if } n = 2m-1, \\ 0 & \text{if } n = 2m. \end{cases} \end{aligned}$$

Therefore

$$C(x) = \frac{L^3}{2} - \frac{48L^3}{\pi^4} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \cos \frac{(2n-1)\pi x}{L}.$$

Example 11.3.6 Find the Fourier sine expansion of $f(x) = x(x^2 - 3Lx + 2L^2)$ on $[0, L]$.

Solution Since $f(0) = f(L) = 0$ and $f''(x) = 6(x - L)$, we see from (11.3.5) that

$$\begin{aligned} b_n &= -\frac{12L}{n^2\pi^2} \int_0^L (x - L) \sin \frac{n\pi x}{L} dx \\ &= \frac{12L^2}{n^3\pi^3} \left[(x - L) \cos \frac{n\pi x}{L} \Big|_0^L - \int_0^L \cos \frac{n\pi x}{L} dx \right] \\ &= \frac{12L^2}{n^3\pi^3} \left[L - \frac{L}{n\pi} \sin \frac{n\pi x}{L} \Big|_0^L \right] = \frac{12L^3}{n^3\pi^3}. \end{aligned}$$

Therefore

$$S(x) = \frac{12L^3}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin \frac{n\pi x}{L}.$$

Example 11.3.7 Find the mixed Fourier cosine expansion of $f(x) = 3x^3 - 4Lx^2 + L^3$ on $[0, L]$.

Solution Since $f'(0) = f(L) = 0$ and $f''(x) = 2(9x - 4L)$, we see from (11.3.6) that

$$\begin{aligned} c_n &= -\frac{16L}{(2n-1)^2\pi^2} \int_0^L (9x - 4L) \cos \frac{(2n-1)\pi x}{2L} dx \\ &= -\frac{32L^2}{(2n-1)^3\pi^3} \left[(9x - 4L) \sin \frac{(2n-1)\pi x}{2L} \Big|_0^L - 9 \int_0^L \sin \frac{(2n-1)\pi x}{2L} dx \right] \\ &= -\frac{32L^2}{(2n-1)^3\pi^3} \left[(-1)^{n+1} 5L + \frac{18L}{(2n-1)\pi} \cos \frac{(2n-1)\pi x}{2L} \Big|_0^L \right] \\ &= \frac{32L^3}{(2n-1)^3\pi^3} \left[(-1)^n 5 + \frac{18}{(2n-1)\pi} \right]. \end{aligned}$$

Therefore

$$C_M(x) = \frac{32L^3}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \left[(-1)^n 5 + \frac{18}{(2n-1)\pi} \right] \cos \frac{(2n-1)\pi x}{2L}.$$

Example 11.3.8 Find the mixed Fourier sine expansion of

$$f(x) = x(2x^2 - 9Lx + 12L^2)$$

on $[0, L]$.

Solution Since $f(0) = f'(L) = 0$, and $f''(x) = 6(2x - 3L)$, we see from (11.3.7) that

$$\begin{aligned} d_n &= -\frac{48L}{(2n-1)^2\pi^2} \int_0^L (2x - 3L) \sin \frac{(2n-1)\pi x}{2L} dx \\ &= \frac{96L^2}{(2n-1)^3\pi^3} \left[(2x - 3L) \cos \frac{(2n-1)\pi x}{2L} \Big|_0^L - 2 \int_0^L \cos \frac{(2n-1)\pi x}{2L} dx \right] \\ &= \frac{96L^2}{(2n-1)^3\pi^3} \left[3L - \frac{4L}{(2n-1)\pi} \sin \frac{(2n-1)\pi x}{2L} \Big|_0^L \right] \\ &= \frac{96L^3}{(2n-1)^3\pi^3} \left[3 + (-1)^n \frac{4}{(2n-1)\pi} \right]. \end{aligned}$$

Therefore

$$S_M(x) = \frac{96L^3}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \left[3 + (-1)^n \frac{4}{(2n-1)\pi} \right] \sin \frac{(2n-1)\pi x}{2L}.$$

Section 12.1

Example 12.1.1 Solve (12.1.4) with $f(x) = x(x^2 - 3Lx + 2L^2)$.

Solution From Example 11.3.6, the Fourier sine series of f on $[0, L]$ is

$$S(x) = \frac{12L^3}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin \frac{n\pi x}{L}.$$

Therefore

$$u(x, t) = \frac{12L^3}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} e^{-n^2\pi^2 a^2 t/L^2} \sin \frac{n\pi x}{L}. \quad \blacksquare$$

Example 12.1.2 Solve (12.1.6) with $f(x) = x$.

Solution From Example 11.3.1, the Fourier cosine series of f on $[0, L]$ is

$$C(x) = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{L}.$$

Therefore

$$u(x, t) = \frac{L}{2} - \frac{4L}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} e^{-(2n-1)^2\pi^2 a^2 t/L^2} \cos \frac{(2n-1)\pi x}{L}. \quad \blacksquare$$

Example 12.1.3 Solve (12.1.7) with $f(x) = x$.

Solution From Example 11.3.4, the mixed Fourier sine series of f on $[0, L]$ is

$$S_M(x) = -\frac{8L}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{2L}.$$

Therefore

$$u(x, t) = -\frac{8L}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)^2} e^{-(2n-1)^2\pi^2 a^2 t/4L^2} \sin \frac{(2n-1)\pi x}{2L}. \quad \blacksquare$$

Figure 12.1.2 shows a graph of $u = u(x, t)$ plotted with respect to x for various values of t . The line $y = x$ corresponds to $t = 0$. The other curves correspond to positive values of t . As t increases, the graphs approach the line $u = 0$.

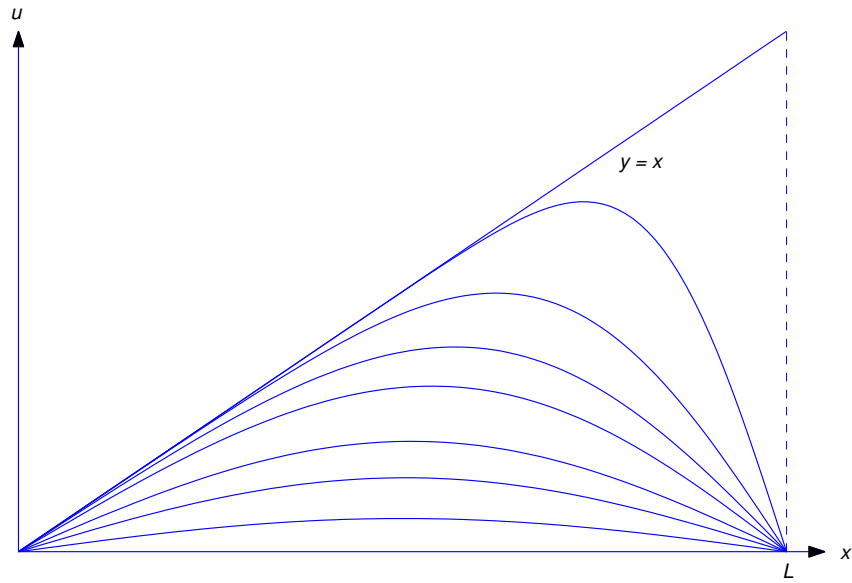


Figure 12.1.2

Example 12.1.4 Solve (12.1.8) with $f(x) = x - L$.

Solution From Example 11.3.3, the mixed Fourier cosine series of f on $[0, L]$ is

$$C_M(x) = -\frac{8L}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{2L}.$$

Therefore

$$u(x, t) = -\frac{8L}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} e^{-(2n-1)^2 \pi^2 a^2 t / 4L^2} \cos \frac{(2n-1)\pi x}{2L}.$$

Example 12.1.5 Solve

$$\begin{aligned} u_t &= u_{xx} - 2, & 0 < x < 1, & \quad t > 0, \\ u(0, t) &= -1, & u(1, t) &= 1, & \quad t > 0, \\ u(x, 0) &= x^3 - 2x^2 + 3x - 1, & 0 \leq x \leq 1. \end{aligned}$$

Solution We leave it to you to show that

$$q(x) = x^2 + x - 1$$

satisfies

$$q'' - 2 = 0, \quad q(0) = -1, \quad q(1) = 1.$$

Therefore

$$u(x, t) = v(x, t) + x^2 + x - 1,$$

where

$$v_t = v_{xx}, \quad 0 < x < 1, \quad t > 0,$$

$$v(0, t) = 0, \quad v(1, t) = 0, \quad t > 0,$$

and

$$v(x, 0) = x^3 - 2x^2 + 3x - 1 - x^2 - x + 1 = x(x^2 - 3x + 2).$$

From Example 12.1.1 with $a = 1$ and $L = 1$,

$$v(x, t) = \frac{12}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} e^{-n^2 \pi^2 t} \sin n\pi x.$$

Therefore

$$u(x, t) = x^2 + x - 1 + \frac{12}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} e^{-n^2 \pi^2 t} \sin n\pi x. \quad \blacksquare$$

Section 12.2

Example 12.2.1 Solve (12.2.1) with

$$f(x) = x(x^3 - 2Lx^2 + L^3) \quad \text{and} \quad g(x) = x(L - x).$$

Solution We leave it to you to verify that f and g satisfy the assumptions of Theorem 12.2.4.

From Exercise 11.3.11.3.39,

$$S_f(x) = \frac{96L^4}{\pi^5} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^5} \sin \frac{(2n-1)\pi x}{L}.$$

From Exercise 11.3.11.3.36,

$$S_g(x) = \frac{8L^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \sin \frac{(2n-1)\pi x}{L}.$$

From (12.2.8),

$$\begin{aligned} u(x, t) = & \frac{96L^4}{\pi^5} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^5} \cos \frac{(2n-1)\pi at}{L} \sin \frac{(2n-1)\pi x}{L} \\ & + \frac{8L^3}{a\pi^4} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^4} \sin \frac{(2n-1)\pi at}{L} \sin \frac{(2n-1)\pi x}{L}. \end{aligned}$$

Theorem 12.1.2 implies that u_{xx} and u_{tt} can be obtained by term by term differentiation, for all (x, t) , so $u_{tt} = a^2 u_{xx}$ for all (x, t) (Exercise 12.2.62). Moreover, Theorem 11.3.2 implies that $S_f(x) = f(x)$ and $S_g(x) = g(x)$ if $0 \leq x \leq L$. Therefore $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$ if $0 \leq x \leq L$. Hence, u is an actual solution of the initial-boundary value problem.

Section 12.3

Example 12.3.1 Define the formal solution of

$$\begin{aligned} u_{xx} + u_{yy} &= 0, & 0 < x < a, & \quad 0 < y < b, \\ u(x, 0) &= f(x), & u(x, b) &= 0, & \quad 0 \leq x \leq a, \\ u(0, y) &= 0, & u(a, y) &= 0, & \quad 0 \leq y \leq b \end{aligned} \quad (12.3.4)$$

(Figure 12.3.4).

Solution The boundary conditions in (12.3.4) require products $v(x, y) = X(x)Y(y)$ such that $X(0) = X(a) = Y(b) = 0$; hence, we let $k = -\lambda$ in (12.3.3). Thus, X and Y must satisfy

$$X'' + \lambda X = 0, \quad X(0) = 0, \quad X(a) = 0 \quad (12.3.5)$$

and

$$Y'' - \lambda Y = 0, \quad Y(b) = 0. \quad (12.3.6)$$

From Theorem 11.1.2, the eigenvalues of (12.3.5) are $\lambda_n = n^2\pi^2/a^2$, with associated eigenfunctions

$$X_n = \sin \frac{n\pi x}{a}, \quad n = 1, 2, 3, \dots$$

Substituting $\lambda = n^2\pi^2/a^2$ into (12.3.6) yields

$$Y'' - (n^2\pi^2/a^2)Y = 0, \quad Y(b) = 0,$$

so we could take

$$Y_n = \sinh \frac{n\pi(b-y)}{a}; \quad (12.3.7)$$

however, because of the nonhomogeneous Dirichlet condition at $y = 0$, it's better to require that $Y_n(0) = 1$, which can be achieved by dividing the right side of (12.3.7) by its value at $y = 0$; thus, we take

$$Y_n = \frac{\sinh n\pi(b-y)/a}{\sinh n\pi b/a}.$$

Then

$$v_n(x, y) = X_n(x)Y_n(y) = \frac{\sinh n\pi(b-y)/a}{\sinh n\pi b/a} \sin \frac{n\pi x}{a},$$

so $v_n(x, 0) = \sin n\pi x/a$ and v_n satisfies (12.3.4) with $f(x) = \sin n\pi x/a$. More generally, if $\alpha_1, \dots, \alpha_m$ are arbitrary constants then

$$u_m(x, y) = \sum_{n=1}^m \alpha_n \frac{\sinh n\pi(b-y)/a}{\sinh n\pi b/a} \sin \frac{n\pi x}{a}$$

satisfies (12.3.4) with

$$f(x) = \sum_{n=1}^m \alpha_n \sin \frac{n\pi x}{a}.$$

Therefore, if f is an arbitrary piecewise smooth function on $[0, a]$, we define the formal solution of (12.3.4) to be

$$u(x, y) = \sum_{n=1}^{\infty} \alpha_n \frac{\sinh n\pi(b-y)/a}{\sinh n\pi b/a} \sin \frac{n\pi x}{a}, \quad (12.3.8)$$

where

$$S(x) = \sum_{n=1}^{\infty} \alpha_n \sin \frac{n\pi x}{a}$$

is the Fourier sine series of f on $[0, a]$; that is,

$$\alpha_n = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a} dx, \quad n = 1, 2, 3, \dots \quad \blacksquare$$

If $y < b$ then

$$\frac{\sinh n\pi(b-y)/a}{\sinh n\pi b/a} \approx e^{-n\pi y/a} \quad (12.3.9)$$

for large n , so the series in (12.3.8) converges if $0 < y < b$; moreover, since also

$$\frac{\cosh n\pi(b-y)/a}{\sinh n\pi b/a} \approx e^{-n\pi y/a}$$

for large n , Theorem 12.1.2 applied twice with $z = x$ and twice with $z = y$ shows that u_{xx} and u_{yy} can be obtained by differentiating u term by term if $0 < y < b$. (Exercise 12.3.37). Therefore u satisfies Laplace's equation in the interior of the rectangle in Figure 12.3.4. Moreover, the series in (12.3.8) also converges on the boundary of the rectangle, and satisfies the three homogeneous boundary conditions in (12.3.4). Therefore, since $u(x, 0) = S(x)$ for $0 \leq x \leq a$, u is an actual solution of (12.3.4) if and only if $S(x) = f(x)$ for $0 \leq x \leq a$. From Theorem 11.3.2, this is true if f is continuous and piecewise smooth on $[0, a]$, and $f(0) = f(a) = 0$.

Example 12.3.2 Solve (12.3.4) with $f(x) = x(x^2 - 3ax + 2a^2)$.

Solution From Example 11.3.6,

$$S(x) = \frac{12a^3}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{n^3} \sin \frac{n\pi x}{a}.$$

Therefore

$$u(x, y) = \frac{12a^3}{\pi^3} \sum_{n=1}^{\infty} \frac{\sinh n\pi(b-y)/a}{n^3 \sinh n\pi b/a} \sin \frac{n\pi x}{a}. \quad (12.3.10)$$

To compute approximate values of $u(x, y)$, we must use partial sums of the form

$$u_m(x, y) = \frac{12a^3}{\pi^3} \sum_{n=1}^m \frac{\sinh n\pi(b-y)/a}{n^3 \sinh n\pi b/a} \sin \frac{n\pi x}{a}.$$

Because of (12.3.9), small values of m provide sufficient accuracy for most applications if $0 < y < b$. Moreover, the n^3 in the denominator in (12.3.10) ensures that this is also true for $y = 0$. For graphing purposes, we chose $a = 2$, $b = 1$, and $m = 10$. Figure 12.3.5 shows the surface

$$u = u(x, y), \quad 0 \leq x \leq 2, \quad 0 \leq y \leq 1,$$

while Figure 12.3.6 shows the curves

$$u = u(x, 0.1k), \quad 0 \leq x \leq 2, \quad k = 0, 1, \dots, 10.$$

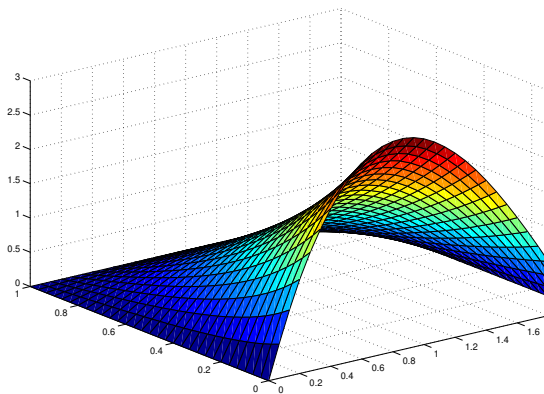


Figure 12.3.5

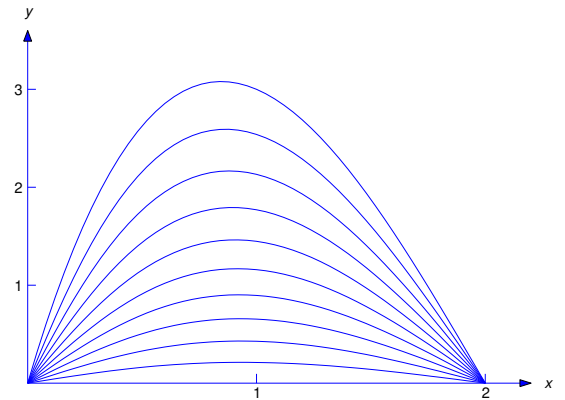


Figure 12.3.6

Example 12.3.3 Define the formal solution of

$$\begin{aligned} u_{xx} + u_{yy} &= 0, & 0 < x < a, & \quad 0 < y < b, \\ u(x, 0) &= 0, & u_y(x, b) &= f(x), & \quad 0 \leq x \leq a, \\ u_x(0, y) &= 0, & u_x(a, y) &= 0, & \quad 0 \leq y \leq b \end{aligned} \quad (12.3.11)$$

(Figure 12.3.7).

Solution The boundary conditions in (12.3.11) require products $v(x, y) = X(x)Y(y)$ such that $X'(0) = X'(a) = Y(0) = 0$; hence, we let $k = -\lambda$ in (12.3.3). Thus, X and Y must satisfy

$$X'' + \lambda X = 0, \quad X'(0) = 0, \quad X'(a) = 0 \quad (12.3.12)$$

and

$$Y'' - \lambda Y = 0, \quad Y(0) = 0. \quad (12.3.13)$$

From Theorem 11.1.3, the eigenvalues of (12.3.12) are $\lambda = 0$, with associated eigenfunction $X_0 = 1$, and $\lambda_n = n^2\pi^2/a^2$, with associated eigenfunctions

$$X_n = \cos \frac{n\pi x}{a}, \quad n = 1, 2, 3, \dots$$

Since $Y_0 = y$ satisfies (12.3.13) with $\lambda = 0$, we take $v_0(x, y) = X_0(x)Y_0(y) = y$. Substituting $\lambda = n^2\pi^2/a^2$ into (12.3.13) yields

$$Y'' - (n^2\pi^2/a^2)Y = 0, \quad Y(0) = 0,$$

so we could take

$$Y_n = \sinh \frac{n\pi y}{a}. \quad (12.3.14)$$

However, because of the nonhomogeneous Neumann condition at $y = b$, it's better to require that $Y'_n(b) = 1$, which can be achieved by dividing the right side of (12.3.14) by the value of its derivative at $y = b$; thus,

$$Y_n = \frac{a \sinh n\pi y/a}{n\pi \cosh n\pi b/a}.$$

Then

$$v_n(x, y) = X_n(x)Y_n(y) = \frac{a \sinh n\pi y/a}{n\pi \cosh n\pi b/a} \cos \frac{n\pi x}{a},$$

so

$$\frac{\partial v_n}{\partial y}(x, b) = \cos \frac{n\pi x}{a}.$$

Therefore v_n satisfies (12.3.11) with $f(x) = \cos n\pi x/a$. More generally, if $\alpha_0, \dots, \alpha_m$ are arbitrary constants then

$$u_m(x, y) = \alpha_0 y + \frac{a}{\pi} \sum_{n=1}^m \alpha_n \frac{\sinh n\pi y/a}{n \cosh n\pi b/a} \cos \frac{n\pi x}{a}$$

satisfies (12.3.11) with

$$f(x) = \alpha_0 + \sum_{n=1}^m \alpha_n \cos \frac{n\pi x}{a}.$$

Therefore, if f is an arbitrary piecewise smooth function on $[0, a]$ we define the formal solution of (12.3.11) to be

$$u(x, y) = \alpha_0 y + \frac{a}{\pi} \sum_{n=1}^{\infty} \alpha_n \frac{\sinh n\pi y/a}{n \cosh n\pi b/a} \cos \frac{n\pi x}{a},$$

where

$$C(x) = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos \frac{n\pi x}{a}$$

is the Fourier cosine series of f on $[0, a]$; that is,

$$\alpha_0 = \frac{1}{a} \int_0^a f(x) dx \quad \text{and} \quad \alpha_n = \frac{2}{a} \int_0^a f(x) \cos \frac{n\pi x}{a} dx, \quad n = 1, 2, 3, \dots$$

Example 12.3.4 Solve (12.3.11) with $f(x) = x$.

Solution From Example 11.3.1,

$$C(x) = \frac{a}{2} - \frac{4a}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{a}.$$

Therefore

$$u(x, y) = \frac{ay}{2} - \frac{4a^2}{\pi^3} \sum_{n=1}^{\infty} \frac{\sinh(2n-1)\pi y/a}{(2n-1)^3 \cosh(2n-1)\pi b/a} \cos \frac{(2n-1)\pi x}{a}. \quad (12.3.15)$$

For graphing purposes, we chose $a = 2$, $b = 1$, and retained the terms through $n = 10$ in (12.3.15). Figure 12.3.8 shows the surface

$$u = u(x, y), \quad 0 \leq x \leq 2, \quad 0 \leq y \leq 1,$$

while Figure 12.3.9 shows the curves

$$u = u(x, .1k), \quad 0 \leq x \leq 2, \quad k = 0, 1, \dots, 10.$$

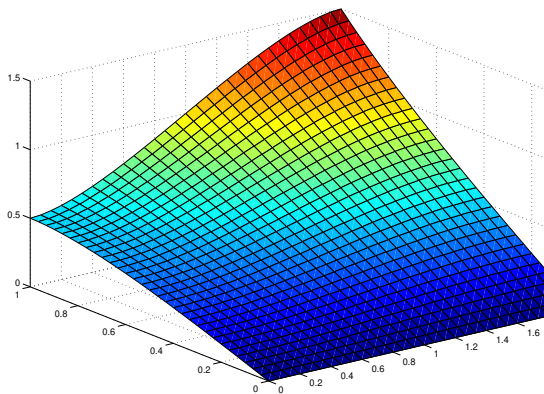


Figure 12.3.8

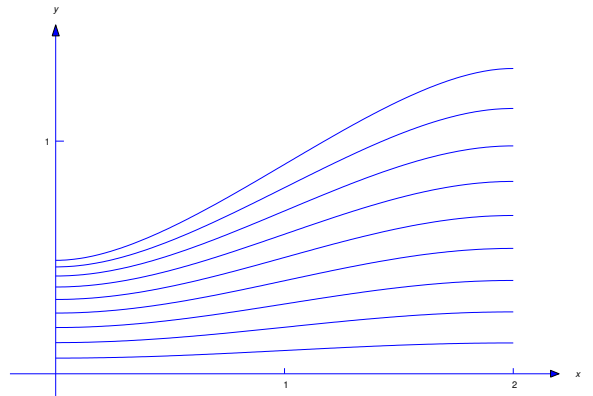


Figure 12.3.9

Example 12.3.5 Define the formal solution of

$$\begin{aligned} u_{xx} + u_{yy} &= 0, & 0 < x < a, & \quad 0 < y < b, \\ u(x, 0) &= 0, & u_y(x, b) &= 0, & \quad 0 \leq x \leq a, \\ u(0, y) &= g(y), & u_x(a, y) &= 0, & \quad 0 \leq y \leq b \end{aligned} \quad (12.3.16)$$

(Figure 12.3.10).

Solution The boundary conditions in (12.3.16) require products $v(x, y) = X(x)Y(y)$ such that $Y(0) = Y'(b) = X'(a) = 0$; hence, we let $k = \lambda$ in (12.3.3). Thus, X and Y must satisfy

$$X'' - \lambda X = 0, \quad X'(a) = 0 \quad (12.3.17)$$

and

$$Y'' + \lambda Y = 0, \quad Y(0) = 0, \quad Y'(b) = 0. \quad (12.3.18)$$

From Theorem 11.1.4, the eigenvalues of (12.3.18) are $\lambda_n = (2n - 1)^2\pi^2/4b^2$, with associated eigenfunctions

$$Y_n = \sin \frac{(2n - 1)\pi y}{2b}, \quad n = 1, 2, 3, \dots$$

Substituting $\lambda = (2n - 1)^2\pi^2/4b^2$ into (12.3.17) yields

$$X'' - ((2n - 1)^2\pi^2/4b^2)X = 0, \quad X'(a) = 0,$$

so we could take

$$X_n = \cosh \frac{(2n - 1)\pi(x - a)}{2b}. \quad (12.3.19)$$

However, because of the nonhomogeneous Dirichlet condition at $x = 0$, it's better to require that $X_n(0) = 1$, which can be achieved by dividing the right side of (12.3.19) by its value at $x = 0$; thus,

$$X_n = \frac{\cosh(2n - 1)\pi(x - a)/2b}{\cosh(2n - 1)\pi a/2b}.$$

Then

$$v_n(x, y) = X_n(x)Y_n(y) = \frac{\cosh(2n - 1)\pi(x - a)/2b}{\cosh(2n - 1)\pi a/2b} \sin \frac{(2n - 1)\pi y}{2b},$$

so

$$v_n(0, y) = \sin \frac{(2n - 1)\pi y}{2b}.$$

Therefore v_n satisfies (12.3.16) with $g(y) = \sin(2n - 1)\pi y/2b$. More generally, if $\alpha_1, \dots, \alpha_m$ are arbitrary constants then

$$u_m(x, y) = \sum_{n=1}^m \alpha_n \frac{\cosh(2n - 1)\pi(x - a)/2b}{\cosh(2n - 1)\pi a/2b} \sin \frac{(2n - 1)\pi y}{2b}$$

satisfies (12.3.16) with

$$g(y) = \sum_{n=1}^m \alpha_n \sin \frac{(2n - 1)\pi y}{2b}.$$

Thus, if g is an arbitrary piecewise smooth function on $[0, b]$, we define the formal solution of (12.3.16) to be

$$u(x, y) = \sum_{n=1}^{\infty} \alpha_n \frac{\cosh(2n - 1)\pi(x - a)/2b}{\cosh(2n - 1)\pi a/2b} \sin \frac{(2n - 1)\pi y}{2b},$$

where

$$S_M(y) = \sum_{n=1}^{\infty} \alpha_n \sin \frac{(2n-1)\pi y}{2b}$$

is the mixed Fourier sine series of g on $[0, b]$; that is,

$$\alpha_n = \frac{2}{b} \int_0^b g(y) \sin \frac{(2n-1)\pi y}{2b} dy.$$

Example 12.3.6 Solve (12.3.16) with $g(y) = y(2y^2 - 9by + 12b^2)$.

Solution From Example 11.3.8,

$$S_M(y) = \frac{96b^3}{\pi^3} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} \left[3 + (-1)^n \frac{4}{(2n-1)\pi} \right] \sin \frac{(2n-1)\pi y}{2b}.$$

Therefore

$$u(x, y) = \frac{96b^3}{\pi^3} \sum_{n=1}^{\infty} \frac{\cosh(2n-1)\pi(x-a)/2b}{(2n-1)^3 \cosh(2n-1)\pi a/2b} \left[3 + (-1)^n \frac{4}{(2n-1)\pi} \right] \sin \frac{(2n-1)\pi y}{2b}.$$

Example 12.3.7 Define the formal solution of

$$\begin{aligned} u_{xx} + u_{yy} &= 0, & 0 < x < a, & \quad 0 < y < b, \\ u_y(x, 0) &= 0, & u(x, b) &= 0, & \quad 0 \leq x \leq a, \\ u_x(0, y) &= 0, & u_x(a, y) &= g(y), & \quad 0 \leq y \leq b \end{aligned} \quad (12.3.20)$$

(Figure 12.3.11).

Solution The boundary conditions in (12.3.20) require products $v(x, y) = X(x)Y(y)$ such that $Y'(0) = Y(b) = X'(0) = 0$; hence, we let $k = \lambda$ in (12.3.3). Thus, X and Y must satisfy

$$X'' - \lambda X = 0, \quad X'(0) = 0 \quad (12.3.21)$$

and

$$Y'' + \lambda Y = 0, \quad Y'(0) = 0, \quad Y(b) = 0. \quad (12.3.22)$$

From Theorem 11.1.4, the eigenvalues of (12.3.22) are $\lambda_n = (2n-1)^2\pi^2/4b^2$, with associated eigenfunctions

$$Y_n = \cos \frac{(2n-1)\pi y}{2b}, \quad n = 1, 2, 3, \dots$$

Substituting $\lambda = (2n-1)^2\pi^2/4b^2$ into (12.3.21) yields

$$X'' - ((2n-1)^2\pi^2/4b^2)X = 0, \quad X'(0) = 0,$$

so we could take

$$X_n = \cosh \frac{(2n-1)\pi x}{2b}. \quad (12.3.23)$$

However, because of the nonhomogeneous Neumann condition at $x = a$, it's better to require that $X'_n(a) = 1$, which can be achieved by dividing the right side of (12.3.23) by the value of its derivative at $x = a$; thus,

$$X_n = \frac{2b \cosh(2n-1)\pi x/2b}{(2n-1)\pi \sinh(2n-1)\pi a/2b}.$$

Then

$$v_n(x, y) = X_n(x)Y_n(y) = \frac{2b \cosh(2n-1)\pi x/2b}{(2n-1)\pi \sinh(2n-1)\pi a/2b} \cos \frac{(2n-1)\pi y}{2b},$$

so

$$\frac{\partial v_n}{\partial x}(a, y) = \cos \frac{(2n-1)\pi y}{2b}.$$

Therefore v_n satisfies (12.3.20) with $g(y) = \cos(2n-1)\pi y/2b$. More generally, if $\alpha_1, \dots, \alpha_m$ are arbitrary constants then

$$u_m(x, y) = \frac{2b}{\pi} \sum_{n=1}^m \alpha_n \frac{\cosh(2n-1)\pi x/2b}{(2n-1) \sinh(2n-1)\pi a/2b} \cos \frac{(2n-1)\pi y}{2b}$$

satisfies (12.3.20) with

$$g(y) = \sum_{n=1}^m \alpha_n \cos \frac{(2n-1)\pi y}{2b}.$$

Therefore, if g is an arbitrary piecewise smooth function on $[0, b]$, we define the formal solution of (12.3.20) to be

$$u(x, y) = \frac{2b}{\pi} \sum_{n=1}^{\infty} \alpha_n \frac{\cosh(2n-1)\pi x/2b}{(2n-1) \sinh(2n-1)\pi a/2b} \cos \frac{(2n-1)\pi y}{2b},$$

where

$$C_M(y) = \sum_{n=1}^{\infty} \alpha_n \cos \frac{(2n-1)\pi y}{2b}$$

is the mixed Fourier cosine series of g on $[0, b]$; that is,

$$\alpha_n = \frac{2}{b} \int_0^b g(y) \cos \frac{(2n-1)\pi y}{2b} dy.$$

Example 12.3.8 Solve (12.3.20) with $g(y) = y - b$.

Solution From Example 11.3.3,

$$C_M(y) = -\frac{8b}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi y}{2b}.$$

Therefore

$$u(x, y) = -\frac{16b^2}{\pi^3} \sum_{n=1}^{\infty} \frac{\cosh(2n-1)\pi x/2b}{(2n-1)^3 \sinh(2n-1)\pi a/2b} \cos \frac{(2n-1)\pi y}{2b}. \quad \blacksquare$$

Example 12.3.9 Define the bounded formal solution of (12.3.24).

Solution Proceeding as in the solution of Example 12.3.1, we find that the building block functions are of the form

$$v_n(x, y) = Y_n(y) \sin \frac{n\pi x}{a},$$

where

$$Y_n'' - (n^2\pi^2/a^2)Y_n = 0.$$

Therefore

$$Y_n = c_1 e^{n\pi y/a} + c_2 e^{-n\pi y/a}$$

where c_1 and c_2 are constants. Although the boundary conditions in (12.3.24) don't restrict c_1 , and c_2 , we must set $c_1 = 0$ to ensure that Y_n is bounded. Letting $c_2 = 1$ yields

$$v_n(x, y) = e^{-n\pi y/a} \sin \frac{n\pi x}{a},$$

and we define the bounded formal solution of (12.3.24) to be

$$u(x, y) = \sum_{n=1}^{\infty} b_n e^{-n\pi y/a} \sin \frac{n\pi x}{a},$$

where

$$S(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a}$$

is the Fourier sine series of f on $[0, a]$. ■

Section 12.4

Example 12.4.1 Find the bounded formal solution of (12.4.2) with $f(\theta) = \theta(\pi^2 - \theta^2)$.

Solution From Example 11.2.6,

$$\theta(\pi^2 - \theta^2) = 12 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^3} \sin n\theta, \quad -\pi \leq \theta \leq \pi,$$

so

$$u(r, \theta) = 12 \sum_{n=1}^{\infty} \frac{r^n}{\rho^n} \frac{(-1)^n}{n^3} \sin n\theta, \quad 0 \leq r \leq \rho, \quad -\pi \leq \theta \leq \pi.$$

Example 12.4.2 Define the formal solution of

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} &= 0, \quad \rho_0 < r < \rho, \quad -\pi \leq \theta < \pi, \\ u(\rho_0, \theta) &= 0, \quad u(\rho, \theta) = f(\theta), \quad -\pi \leq \theta < \pi, \end{aligned} \tag{12.4.9}$$

where $0 < \rho_0 < \rho$ (Figure 12.4.2).

Solution We use separation of variables exactly as before, except that now we choose the constants in (12.4.5) and (12.4.7) so that $R_n(\rho_0) = 0$ for $n = 0, 1, 2, \dots$. In view of the nonhomogeneous Dirichlet condition on the boundary $r = \rho$, it's also convenient to require that $R_n(\rho) = 1$ for $n = 0, 1, 2, \dots$. We leave it to you to verify that

$$R_0(r) = \frac{\ln(r/\rho_0)}{\ln(\rho/\rho_0)} \quad \text{and} \quad R_n = \frac{\rho_0^{-n}r^n - \rho_0^n r^{-n}}{\rho_0^{-n}\rho^n - \rho_0^n \rho^{-n}}, \quad n = 1, 2, 3, \dots$$

satisfy these requirements. Therefore

$$v_0(r, \theta) = \frac{\ln(r/\rho_0)}{\ln(\rho/\rho_0)}$$

and

$$v_n(r, \theta) = \frac{\rho_0^{-n}r^n - \rho_0^n r^{-n}}{\rho_0^{-n}\rho^n - \rho_0^n \rho^{-n}} (\alpha_n \cos n\theta + \beta_n \sin n\theta), \quad n = 1, 2, 3, \dots,$$

where α_n and β_n are arbitrary constants.

If $\alpha_0, \alpha_1, \dots, \alpha_m$ and $\beta_1, \beta_2, \dots, \beta_m$ are arbitrary constants then

$$u_m(r, \theta) = \alpha_0 \frac{\ln(r/\rho_0)}{\ln(\rho/\rho_0)} + \sum_{n=1}^m \frac{\rho_0^{-n}r^n - \rho_0^n r^{-n}}{\rho_0^{-n}\rho^n - \rho_0^n \rho^{-n}} (\alpha_n \cos n\theta + \beta_n \sin n\theta)$$

satisfies (12.4.9), with

$$f(\theta) = \alpha_0 + \sum_{n=1}^m (\alpha_n \cos n\theta + \beta_n \sin n\theta).$$

This motivates us to define the formal solution of (12.4.9) for general f to be

$$u(r, \theta) = \alpha_0 \frac{\ln(r/\rho_0)}{\ln(\rho/\rho_0)} + \sum_{n=1}^{\infty} \frac{\rho_0^{-n}r^n - \rho_0^n r^{-n}}{\rho_0^{-n}\rho^n - \rho_0^n \rho^{-n}} (\alpha_n \cos n\theta + \beta_n \sin n\theta),$$

where

$$F(\theta) = \alpha_0 + \sum_{n=1}^{\infty} (\alpha_n \cos n\theta + \beta_n \sin n\theta)$$

is the Fourier series of f on $[-\pi, \pi]$.

Example 12.4.3 Define the bounded formal solution of

$$\begin{aligned} u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} &= 0, \quad 0 < r < \rho, \quad 0 < \theta < \gamma, \\ u(\rho, \theta) &= f(\theta), \quad 0 \leq \theta \leq \gamma, \\ u(r, 0) &= 0, \quad u(r, \gamma) = 0, \quad 0 < r < \rho, \end{aligned} \tag{12.4.10}$$

where $0 < \gamma < 2\pi$ (Figure 12.4.3).

Solution Now $v(r, \theta) = R(r)\Theta(\theta)$, where

$$r^2 R'' + rR' - \lambda R = 0 \tag{12.4.11}$$

and

$$\Theta'' + \lambda\Theta = 0, \quad \Theta(0) = 0, \quad \Theta(\gamma) = 0. \tag{12.4.12}$$

From Theorem 11.1.2, the eigenvalues of (12.4.12) are $\lambda_n = n^2\pi^2/\gamma^2$, with associated eigenfunction $\Theta_n = \sin n\pi\theta/\gamma$, $n = 1, 2, 3, \dots$. Substituting $\lambda = n^2\pi^2/\gamma^2$ into (12.4.11) yields the Euler equation

$$r^2 R_n'' + r R_n' - \frac{n^2\pi^2}{\gamma^2} R_n = 0.$$

The indicial polynomial of this equation is

$$s(s-1) + s - \frac{n^2\pi^2}{\gamma^2} = \left(s - \frac{n\pi}{\gamma}\right) \left(s + \frac{n\pi}{\gamma}\right),$$

so

$$R_n = c_1 r^{n\pi/\gamma} + c_2 r^{-n\pi/\gamma},$$

by Theorem 7.4.3. To obtain a solution that remains bounded as $r \rightarrow 0+$ we let $c_2 = 0$. Because of the Dirichlet condition at $r = \rho$, it's convenient to have $R_n(\rho) = 1$; therefore we take $c_1 = \rho^{-n\pi/\gamma}$, so

$$R_n(r) = \frac{r^{n\pi/\gamma}}{\rho^{n\pi/\gamma}}.$$

Now

$$v_n(r, \theta) = R_n(r)\Theta_n(\theta) = \frac{r^{n\pi/\gamma}}{\rho^{n\pi/\gamma}} \sin \frac{n\pi\theta}{\gamma}$$

satisfies (12.4.10) with

$$f(\theta) = \sin \frac{n\pi\theta}{\gamma}.$$

More generally, if $\alpha_1, \alpha_2, \dots, \alpha_m$ are arbitrary constants then

$$u_m(r, \theta) = \sum_{n=1}^m \alpha_n \frac{r^{n\pi/\gamma}}{\rho^{n\pi/\gamma}} \sin \frac{n\pi\theta}{\gamma}$$

satisfies (12.4.10) with

$$f(\theta) = \sum_{n=1}^m \alpha_n \sin \frac{n\pi\theta}{\gamma}.$$

This motivates us to define the bounded formal solution of (12.4.10) to be

$$u(r, \theta) = \sum_{n=1}^{\infty} \alpha_n \frac{r^{n\pi/\gamma}}{\rho^{n\pi/\gamma}} \sin \frac{n\pi\theta}{\gamma},$$

where

$$S(\theta) = \sum_{n=1}^{\infty} \alpha_n \sin \frac{n\pi\theta}{\gamma}$$

is the Fourier sine expansion of f on $[0, \gamma]$; that is,

$$\alpha_n = \frac{2}{\gamma} \int_0^\gamma f(\theta) \sin \frac{n\pi\theta}{\gamma} d\theta.$$

Section 13.1

Example 13.1.1 Consider the boundary value problem

$$y'' + y = 1, \quad y(0) = 0, \quad y(\pi/2) = 0.$$

The general solution of $y'' + y = 1$ is

$$y = 1 + c_1 \sin x + c_2 \cos x,$$

so $y(0) = 0$ if and only if $c_2 = -1$ and $y(\pi/2) = 0$ if and only if $c_1 = -1$. Therefore

$$y = 1 - \sin x - \cos x$$

is the unique solution of the boundary value problem.

Example 13.1.2 Consider the boundary value problem

$$y'' + y = 1, \quad y(0) = 0, \quad y(\pi) = 0.$$

Again, the general solution of $y'' + y = 1$ is

$$y = 1 + c_1 \sin x + c_2 \cos x,$$

so $y(0) = 0$ if and only if $c_2 = -1$, but $y(\pi) = 0$ if and only if $c_2 = 1$. Therefore the boundary value problem has no solution.

Example 13.1.3 Consider the boundary value problem

$$y'' + y = \sin 2x, \quad y(0) = 0, \quad y(\pi) = 0.$$

You can use the method of undetermined coefficients (Section 5.5) to find that the general solution of $y'' + y = \sin 2x$ is

$$y = -\frac{\sin 2x}{3} + c_1 \sin x + c_2 \cos x.$$

The boundary conditions $y(0) = 0$ and $y(\pi) = 0$ both require that $c_2 = 0$, but they don't restrict c_1 . Therefore the boundary value problem has infinitely many solutions

$$y = -\frac{\sin 2x}{3} + c_1 \sin x,$$

where c_1 is arbitrary.

Example 13.1.4 Solve the boundary value problem

$$x^2 y'' - 2xy' + 2y - 2x^3 = 0, \quad y(1) = 4, \quad y'(2) = 3, \quad (13.1.13)$$

given that $\{x, x^2\}$ is a fundamental set of solutions of the complementary equation.

Solution Using variation of parameters (Section 5.7), you can show that $y_p = x^3$ is a solution of the nonhomogeneous equation

$$x^2 y'' - 2xy' + 2y = 2x^3.$$

Therefore the solution of (13.1.13) can be written as

$$y = x^3 + c_1 x + c_2 x^2.$$

Then

$$y' = 3x^2 + c_1 + 2c_2 x,$$

and imposing the boundary conditions yields the system

$$\begin{aligned} c_1 + c_2 &= 3 \\ c_1 + 4c_2 &= -9, \end{aligned}$$

so $c_1 = 7$ and $c_2 = -4$. Therefore

$$y = x^3 + 7x - 4x^2$$

is the unique solution of (13.1.13).

Example 13.1.5 Solve the boundary value problem

$$y'' - 7y' + 12y = 4e^{2x}, \quad y(0) = 3, \quad y(1) = 5e^2.$$

Solution From Example 5.4.1, $y_p = 2e^{2x}$ is a particular solution of

$$y'' - 7y' + 12y = 4e^{2x}. \quad (13.1.14)$$

Since $\{e^{3x}, e^{4x}\}$ is a fundamental set for the complementary equation, we could write the solution of (13.1.14) as

$$y = 2e^{2x} + c_1 e^{3x} + c_2 e^{4x}$$

and determine c_1 and c_2 by imposing the boundary conditions. However, this would lead to some tedious algebra, and the form of the solution would be very unappealing. (Try it!) In this case it's convenient to use the fundamental system $\{z_1, z_2\}$ mentioned in Theorem 13.1.2(e); that is, we choose $\{z_1, z_2\}$ so that $B_1(z_1) = z_1(0) = 0$ and $B_2(z_2) = z_2(1) = 0$. It is easy to see that

$$z_1 = e^{3x} - e^{4x} \quad \text{and} \quad z_2 = e^{3(x-1)} - e^{4(x-1)}$$

satisfy these requirements. Now we write the solution of (13.1.14) as

$$y = 2e^{2x} + c_1 (e^{3x} - e^{4x}) + c_2 (e^{3(x-1)} - e^{4(x-1)}).$$

Imposing the boundary conditions $y(0) = 3$ and $y(1) = 5e^2$ yields

$$3 = 2 + c_2 e^{-4}(e - 1) \quad \text{and} \quad 5e^2 = 2e^2 + c_1 e^3(1 - e).$$

Therefore

$$c_1 = \frac{3}{e(1 - e)}, \quad c_2 = \frac{e^4}{e - 1},$$

and

$$y = 2e^{2x} + \frac{3}{e(1 - e)}(e^{3x} - e^{4x}) + \frac{e^4}{e - 1}(e^{3(x-1)} - e^{4(x-1)}).$$

Example 13.1.6 Solve the boundary value problem

$$y'' + y = F(x), \quad y(0) + y'(0) = 0, \quad y(\pi) - y'(\pi) = 0, \quad (13.1.21)$$

and find the Green's function for this problem.

Solution Here

$$B_1(y) = y(0) + y'(0) \quad \text{and} \quad B_2(y) = y(\pi) - y'(\pi).$$

Let $\{z_1, z_2\} = \{\cos x, \sin x\}$, which is a fundamental set of solutions of $y'' + y = 0$. Then

$$B_1(z_1) = (\cos x - \sin x)|_{x=0} = 1$$

$$B_2(z_1) = (\cos x + \sin x)|_{x=\pi} = -1$$

and

$$B_1(z_2) = (\sin x + \cos x)|_{x=0} = 1$$

$$B_2(z_2) = (\sin x - \cos x)|_{x=\pi} = 1.$$

Therefore

$$B_1(z_1)B_2(z_2) - B_1(z_2)B_2(z_1) = 2,$$

so Theorem 13.1.2 implies that (13.1.21) has a unique solution. Let

$$y_1 = B_1(z_2)z_1 - B_1(z_1)z_2 = \cos x - \sin x$$

and

$$y_2 = B_2(z_2)z_1 - B_2(z_1)z_2 = \cos x + \sin x.$$

Then $B_1(y_1) = 0$, $B_2(y_2) = 0$, and the Wronskian of $\{y_1, y_2\}$ is

$$W(x) = \begin{vmatrix} \cos x - \sin x & \cos x + \sin x \\ -\sin x - \cos x & -\sin x + \cos x \end{vmatrix} = 2.$$

Since $P_0 = 1$, (13.1.17) yields the solution

$$y(x) = \frac{\cos x - \sin x}{2} \int_x^\pi F(t)(\cos t + \sin t) dt + \frac{\cos x + \sin x}{2} \int_0^x F(t)(\cos t - \sin t) dt.$$

The Green's function is

$$G(x, t) = \begin{cases} \frac{(\cos t - \sin t)(\cos x + \sin x)}{2}, & 0 \leq t \leq x, \\ \frac{(\cos x - \sin x)(\cos t + \sin t)}{2}, & x \leq t \leq \pi. \end{cases}$$

Example 13.1.7 Applying Theorem 13.1.4 to the boundary value problem

$$y'' + y = F(x), \quad y(0) = 0, \quad y(\pi) = 0 \quad (13.1.25)$$

explains the Examples 13.1.2 and 13.1.3. The complementary equation $y'' + y = 0$ has the linearly independent solutions $y_1 = \sin x$ and $y_2 = \cos x$, and y_1 satisfies both boundary conditions. Since $P_0 = 1$ and

$$W = \begin{vmatrix} \sin x & \cos x \\ \cos x & -\sin x \end{vmatrix} = -1,$$

(13.1.23) reduces to

$$\int_0^\pi F(x) \sin x \, dx = 0.$$

From Example 13.1.2, $F(x) = 1$ and

$$\int_0^\pi F(x) \sin x \, dx = \int_0^\pi \sin x \, dx = 2,$$

so Theorem 13.1.4 implies that (13.1.25) has no solution. In Example 13.1.3,

$$F(x) = \sin 2x = 2 \sin x \cos x$$

and

$$\int_0^\pi F(x) \sin x \, dx = 2 \int_0^\pi \sin^2 x \cos x \, dx = \frac{2}{3} \sin^3 x \Big|_0^\pi = 0,$$

so Theorem 13.1.4 implies that (13.1.25) has infinitely many solutions, differing by constant multiples of $y_1(x) = \sin x$.

Section 13.2

Example 13.2.1 Solve the eigenvalue problem

$$y'' + 3y' + 2y + \lambda y = 0, \quad y(0) = 0, \quad y(1) = 0. \quad (13.2.2)$$

Solution The characteristic equation of (13.2.2) is

$$r^2 + 3r + 2 + \lambda = 0,$$

with zeros

$$r_1 = \frac{-3 + \sqrt{1 - 4\lambda}}{2} \quad \text{and} \quad r_2 = \frac{-3 - \sqrt{1 - 4\lambda}}{2}.$$

If $\lambda < 1/4$ then r_1 and r_2 are real and distinct, so the general solution of the differential equation in (13.2.2) is

$$y = c_1 e^{r_1 x} + c_2 e^{r_2 x}.$$

The boundary conditions require that

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 e^{r_1} + c_2 e^{r_2} &= 0. \end{aligned}$$

Since the determinant of this system is $e^{r_2} - e^{r_1} \neq 0$, the system has only the trivial solution. Therefore λ isn't an eigenvalue of (13.2.2).

If $\lambda = 1/4$ then $r_1 = r_2 = -3/2$, so the general solution of (13.2.2) is

$$y = e^{-3x/2}(c_1 + c_2x).$$

The boundary condition $y(0) = 0$ requires that $c_1 = 0$, so $y = c_2xe^{-3x/2}$ and the boundary condition $y(1) = 0$ requires that $c_2 = 0$. Therefore $\lambda = 1/4$ isn't an eigenvalue of (13.2.2).

If $\lambda > 1/4$ then

$$r_1 = -\frac{3}{2} + i\omega \quad \text{and} \quad r_2 = -\frac{3}{2} - i\omega,$$

with

$$\omega = \frac{\sqrt{4\lambda - 1}}{2} \quad \text{or, equivalently,} \quad \lambda = \frac{1 + 4\omega^2}{4}. \quad (13.2.3)$$

In this case the general solution of the differential equation in (13.2.2) is

$$y = e^{-3x/2}(c_1 \cos \omega x + c_2 \sin \omega x).$$

The boundary condition $y(0) = 0$ requires that $c_1 = 0$, so $y = c_2e^{-3x/2} \sin \omega x$. The condition $y(1) = 0$ holds with $c_2 \neq 0$ if and only if $\omega = n\pi$, where n is an integer. We may assume that n is a positive integer. (Why?). From (13.2.3), the eigenvalues are $\lambda_n = (1 + 4n^2\pi^2)/4$, with associated eigenfunctions

$$y_n = e^{-3x/2} \sin n\pi x, \quad n = 1, 2, 3, \dots$$

Example 13.2.2 Solve the eigenvalue problem

$$x^2y'' + xy' + \lambda y = 0, \quad y(1) = 0, \quad y(2) = 0. \quad (13.2.4)$$

Solution If $\lambda = 0$, the differential equation in (13.2.4) reduces to $x(xy')' = 0$, so $xy' = c_1$,

$$y' = \frac{c_1}{x}, \quad \text{and} \quad y = c_1 \ln x + c_2.$$

The boundary condition $y(1) = 0$ requires that $c_2 = 0$, so $y = c_1 \ln x$. The boundary condition $y(2) = 0$ requires that $c_1 \ln 2 = 0$, so $c_1 = 0$. Therefore zero isn't an eigenvalue of (13.2.4).

If $\lambda < 0$, we write $\lambda = -k^2$ with $k > 0$, so (13.2.4) becomes

$$x^2y'' + xy' - k^2y = 0,$$

an Euler equation (Section 7.4) with indicial equation

$$r^2 - k^2 = (r - k)(r + k) = 0.$$

Therefore

$$y = c_1x^k + c_2x^{-k}.$$

The boundary conditions require that

$$\begin{aligned} c_1 + c_2 &= 0 \\ 2^k c_1 + 2^{-k} c_2 &= 0. \end{aligned}$$

Since the determinant of this system is $2^{-k} - 2^k \neq 0$, $c_1 = c_2 = 0$. Therefore (13.2.4) has no negative eigenvalues.

If $\lambda > 0$ we write $\lambda = k^2$ with $k > 0$. Then (13.2.4) becomes

$$x^2 y'' + xy' + k^2 y = 0,$$

an Euler equation with indicial equation

$$r^2 + k^2 = (r - ik)(r + ik) = 0,$$

so

$$y = c_1 \cos(k \ln x) + c_2 \sin(k \ln x).$$

The boundary condition $y(1) = 0$ requires that $c_1 = 0$. Therefore $y = c_2 \sin(k \ln x)$. This holds with $c_2 \neq 0$ if and only if $k = n\pi/\ln 2$, where n is a positive integer. Hence, the eigenvalues of (13.2.4) are $\lambda_n = (n\pi/\ln 2)^2$, with associated eigenfunctions

$$y_n = \sin\left(\frac{n\pi}{\ln 2} \ln x\right), \quad n = 1, 2, 3, \dots$$

Example 13.2.3 Rewrite the eigenvalue problem

$$y'' + 3y' + (2 + \lambda)y = 0, \quad y(0) = 0, \quad y(1) = 0 \quad (13.2.11)$$

of Example 13.2.1 as a Sturm-Liouville problem.

Solution Comparing (13.2.11) to (13.2.7) shows that $u(x) = 3$, so we take $U(x) = 3x$ and $p(x) = e^{3x}$. Multiplying the differential equation in (13.2.11) by e^{3x} yields

$$e^{3x}(y'' + 3y') + 2e^{3x}y + \lambda e^{3x}y = 0.$$

Since

$$e^{3x}(y'' + 3y') = (e^{3x}y')',$$

(13.2.11) is equivalent to the Sturm-Liouville problem

$$(e^{3x}y')' + 2e^{3x}y + \lambda e^{3x}y = 0, \quad y(0) = 0, \quad y(1) = 0. \quad (13.2.12)$$

Example 13.2.4 Rewrite the eigenvalue problem

$$x^2 y'' + xy' + \lambda y = 0, \quad y(1) = 0, \quad y(2) = 0 \quad (13.2.13)$$

of Example 13.2.2 as a Sturm-Liouville problem.

Solution Dividing the differential equation in (13.2.13) by x^2 yields

$$y'' + \frac{1}{x}y' + \frac{\lambda}{x^2}y = 0.$$

Comparing this to (13.2.7) shows that $u(x) = 1/x$, so we take $U(x) = \ln x$ and $p(x) = e^{\ln x} = x$. Multiplying the differential equation by x yields

$$xy'' + y' + \frac{\lambda}{x}y = 0.$$

Since

$$xy'' + y' = (xy')',$$

(13.2.13) is equivalent to the Sturm-Liouville problem

$$(xy')' + \frac{\lambda}{x}y = 0, \quad y(1) = 0, \quad y(2) = 0. \quad (13.2.14)$$

■

Example 13.2.5 Solve the Sturm–Liouville problem

$$y'' + \lambda y = 0, \quad y(0) + y'(0) = 0, \quad y(1) + 3y'(1) = 0. \quad (13.2.15)$$

Solution If $\lambda = 0$, the differential equation in (13.2.15) reduces to $y'' = 0$, with general solution $y = c_1 + c_2x$. The boundary conditions require that

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 + 4c_2 &= 0, \end{aligned}$$

so $c_1 = c_2 = 0$. Therefore zero isn't an eigenvalue of (13.2.15).

If $\lambda < 0$, we write $\lambda = -k^2$ where $k > 0$, and the differential equation in (13.2.15) becomes $y'' - k^2y = 0$, with general solution

$$y = c_1 \cosh kx + c_2 \sinh kx, \quad (13.2.16)$$

so

$$y' = k(c_1 \sinh kx + c_2 \cosh kx).$$

The boundary conditions require that

$$\begin{aligned} c_1 + kc_2 &= 0 \\ (\cosh k + 3k \sinh k)c_1 + (\sinh k + 3k \cosh k)c_2 &= 0. \end{aligned} \quad (13.2.17)$$

The determinant of this system is

$$\begin{aligned} D_N(k) &= \begin{vmatrix} 1 & k \\ \cosh k + 3k \sinh k & \sinh k + 3k \cosh k \end{vmatrix} \\ &= (1 - 3k^2) \sinh k + 2k \cosh k. \end{aligned}$$

Therefore the system (13.2.17) has a nontrivial solution if and only if $D_N(k) = 0$ or, equivalently,

$$\tanh k = -\frac{2k}{1 - 3k^2}. \quad (13.2.18)$$

The graph of the right side (Figure 13.2.1) has a vertical asymptote at $k = 1/\sqrt{3}$. Since the two sides have different signs if $k < 1/\sqrt{3}$, this equation has no solution in $(0, 1/\sqrt{3})$. Figure 13.2.1 shows the graphs of the two sides of (13.2.18) on an interval to the right of the vertical asymptote, which is indicated by the dashed line. You can see that the two curves intersect near $k_0 = 1.2$. Given this estimate, you can use Newton's method to compute k_0 more accurately. We computed $k_0 \approx 1.1219395$. Therefore $-k_0^2 \approx -1.2587483$ is an eigenvalue of (13.2.15). From (13.2.16) and the first equation in (13.2.17),

$$y_0 = k_0 \cosh k_0x - \sinh k_0x.$$

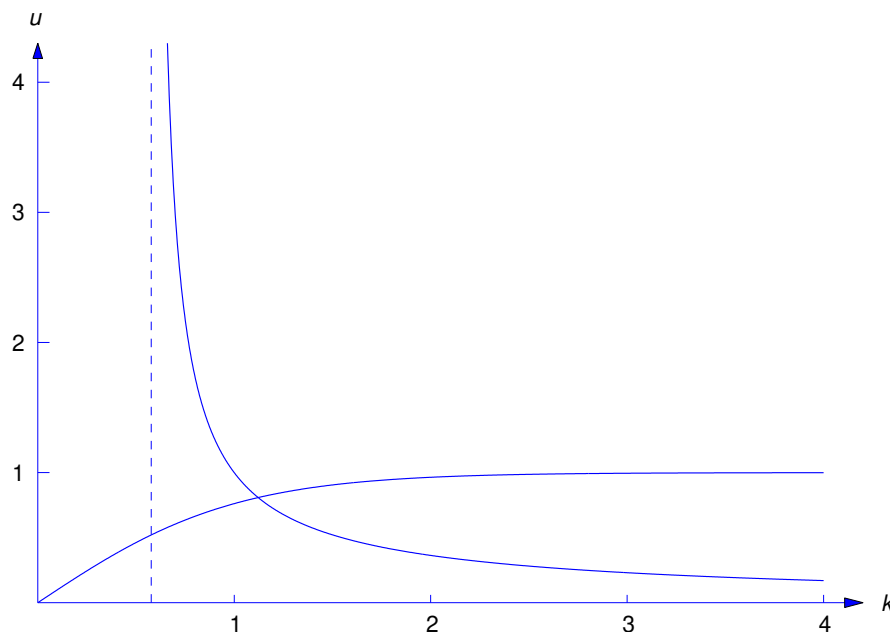


Figure 13.2.1 $u = \tanh k$ and $u = -2k/(1 - 3k^2)$

If $\lambda > 0$ we write $\lambda = k^2$ where $k > 0$, and differential equation in (13.2.15) becomes $y'' + k^2 y = 0$, with general solution

$$y = c_1 \cos kx + c_2 \sin kx, \quad (13.2.19)$$

so

$$y' = k(-c_1 \sin kx + c_2 \cos kx).$$

The boundary conditions require that

$$\begin{aligned} c_1 + kc_2 &= 0 \\ (\cos k - 3k \sin k)c_1 + (\sin k + 3k \cos k)c_2 &= 0. \end{aligned} \quad (13.2.20)$$

The determinant of this system is

$$\begin{aligned} D_P(k) &= \begin{vmatrix} 1 & k \\ \cos k - 3k \sin k & \sin k + 3k \cos k \end{vmatrix} \\ &= (1 + 3k^2) \sin k + 2k \cos k. \end{aligned}$$

The system (13.2.20) has a nontrivial solution if and only if $D_P(k) = 0$ or, equivalently,

$$\tan k = -\frac{2k}{1 + 3k^2}.$$

Figure 13.2.2 shows the graphs of the two sides of this equation. You can see from the figure that the graphs intersect at infinitely many points $k_n \approx n\pi$ ($n = 1, 2, 3, \dots$), where the error in this approximation approaches zero as $n \rightarrow \infty$. Given this estimate, you can use Newton's method to compute k_n more accurately. We computed

$$\begin{aligned} k_1 &\approx 2.9256856, \\ k_2 &\approx 6.1765914, \\ k_3 &\approx 9.3538959, \\ k_4 &\approx 12.5132570. \end{aligned}$$

The estimates of the corresponding eigenvalues $\lambda_n = k_n^2$ are

$$\lambda_1 \approx 8.5596361,$$

$$\lambda_2 \approx 38.1502809,$$

$$\lambda_3 \approx 87.4953676,$$

$$\lambda_4 \approx 156.5815998.$$

From (13.2.19) and the first equation in (13.2.20),

$$y_n = k_n \cos k_n x - \sin k_n x$$

is an eigenfunction associated with λ_n ■

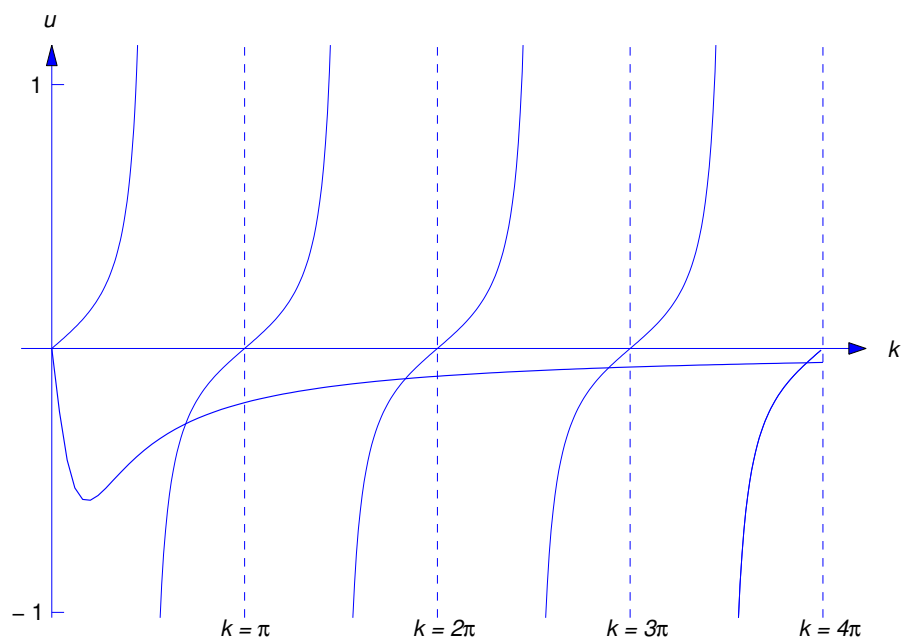


Figure 13.2.2 $u = \tan k$ and $u = -2k/(1 + 3k^2)$